Aggregated threshold functions : a characterization of the world electoral systems between 1945-2000

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Abstract: La tesis de Rubén Ruiz Rufino se titula Aggregated Threshold Functions. A Characterization of the World Electoral Systems between 1945–2000. Fue también dirigida por el profesor Przeworski y fue defendida en la Universidad Complutense de Madrid. Este trabajo constituye una aportación muy importante al estudio de los sistemas electorales, sobre todo por el cálculo de las funciones agregadas de umbrales referidas a todo un sistema electoral, no solamente a un distrito, y para cualquier sistema electoral, de carácter mayoritario, proporcional con fórmulas de divisorios o cuotas, y para sistemas de carácter híbrido. Es importante también por la utilización para esas funciones de un procedimiento de optimización de escaños que atiende a qué combinación de distritos permiten alcanzar un número de escaños en todo el país con el menor porcentaje de votos. La tesis atiende a qué porcentaje más pequeño de votos es necesario para obtener un escaño: es decir, cuál es el criterio de inclusión para pequeños partidos. También a qué porcentaje más pequeño de votos es necesario para obtener el 50% de los escaños: es decir, cuál es el criterio para la formación de mayorías. La investigación se basa en 595 elecciones, entre 1945 y el año 2000, en 102 países y 184 sistemas electorales.

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AGGREGATED THRESHOLD FUNCTIONS. A CHARACTERIZATION OF THE WORLD ELECTORAL SYSTEMS BETWEEN 1945-2000

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Esta obra se presentó como tesis doctoral en el Departamento de Sociología I - Cambio Social - de la Universidad Complutense, el 7 de Junio de 2005. El tribunal estuvo compuesto por los profesores doctores José María Maravall Herrero (Presidente), Julián Santamaría Ossorio (Secretario), Joan Botella Corral, Josep Maria Colomer Calsina y Carlos Flores Juberías. La tesis obtuvo la calificación de Sobresaliente *cum laude*.

Rubén Ruiz Rufino (Granada, 1975) acabó su licenciatura en Derecho en la Universidad de Granada en 1998 después de haber realizado estancias en las universidades de Göttingen (Alemania) y Edimburgo (Reino Unido). En 1999 realizó un *Master* en teoría social y política en la Universidad de Edimburgo (Reino Unido). Entró a formar parte del Centro de Estudios Avanzados en Ciencias Sociales en la XIII promoción de estudiantes y acabó el *Master* en 2001. En los años 2004 y 2005 ha sido visiting student en el departamento de ciencia política de la New York University (USA). Ha realizado su tesis doctoral bajo la dirección del Profesor Adam Przeworski, miembro del comité científico del CEACS.

A mis padres.
A Merche.

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- M_d :Magnitude of district d
- $\mathbf{M}_d$:Vector of distribution of districts
- V_d^p :Share of votes of party p won in district d
- S_d^p :Number of seats obtained by party p in district d
- S_d :Any number of seats in district d.
- P :Total number of competing parties
- P_d:Total number of competing parties in district d
- p :Competing party p
- F :Electoral formula
- L :Largest remainder among all parties with a share of the vote, $V_d^p > 0.5$
- V_d : Total valid votes in district d
- $Q(n)$:Quota of votes
- n :Modifier of the quota
- N :Set of modifiers of the quota
- Z_p :Full quota for party p

- r_p :Remainder of the quota for party p
- R :Number of remaining seats
- X :Divisor
- c :Adjustment term
- D :Number of Districts
- $\mathbf{S}_j$:Vector of distribution of seats among all districts
- $\mathbf{S}_j^*$:Vector of distribution of seats among all districts that produce $\min V_{S_T}^{nec}$
- S_{d_j} :Number of seats in district d in a particular distribution of seats, j
- $\mathbf{S}_j^p$:Vector of distribution of seats among all districts for party p
- $S_{d_j}^p$:Number of seats won by party p in district d in a particular distribution of seats, j
- S_T :Sum of all seats that form the vector $\mathbf{S}_j$
- S_T^p :Sum of all seats that form the vector $\mathbf{S}_j^p$
- M :Size of the Parliament
- $\mathbf{V}^p$:Vector of distribution of votes among districts for party p
- V_d^p :Share of votes for party p in district d
- V_T^p :Total share of votes won by party p in all districts
- $V_{S_d}^{nec}(F, M_{d_i}, P_d)$:Threshold Function of necessary votes to obtain S_d seats

- $V^{suf}_{S_d}(F, M_d, P_d)$:Threshold Function of sufficient votes to obtain S_d seats
- $V^{nec}_{S_T}(P, D, \mathbf{S}_j, F, P)$:Aggregated Threshold Function of necessary votes to obtain S_T seats distributed according to the vector $\mathbf{S}_j$
- $V^{suf}_{S_T}(P, D, \mathbf{S}_j, F, P)$:Aggregated Threshold Function of sufficient votes to obtain S_T seats distributed according to the vector $\mathbf{S}_j$
- DI :Distortion Index
- $DI_{L\&H}$:Distortion Index proposed by Loosemore and Hanby (1971)
- DI_{lsi} :Least Square Index (Gallagher 1991)
- $DI_{d'Hondt}$:Distortion Index based on the d'Hondt electoral formula (Gallagher 1991)
- DI_{S-L} : Distortion Index based on the Sainte-Laguë electoral formula (Gallagher 1991)
- PR : Proportional representation electoral systems
- T_L :Legal threshold
- T_{inc} :Threshold of inclusion as developed by Lijphart (1994)
- T_{exc} :Threshold of exclusion as developed by Lijphart (1994)
- T^{eff}_{Lijp} :Effective threshold as developed by Lijphart (1994)
- $T^{eff}_{T\&S}$:Effective threshold as developed by Taagepera and Shugart (1989)
- T^{eff}_{Taag} :Effective threshold as developed by Taagepera (Lijphart 1994)

It is difficult to write a dissertation without financial and emotional support. This dissertation has been no exception and the debts of gratitude that I have received during this time have been so numerous that they are difficult to condense in a couple of pages. I am very particularly indebted to the Juan March Foundation and its Center for Advanced Studies in the Social Sciences. At this Center I have discovered the true essence of the social sciences through its remarkable teaching. First, during the Masters and then through the constant support of all the staff during my doctoral research. I most warmly thank Andrew Richards and José Ramón Montero for all the time they spent with me when my dissertation was merely a bunch of chaotic intuitions, and for all their illuminating suggestions. I must also thank the staff in the Library who have always been at my total disposition to help me locate books and impossible-to-find articles. My gratitude to Martha Peach, Almudena Knecht, Paz Fernández and Gema Sánchez-Delgado who have patiently listened to all my demands and attended to them in the most efficient way imaginable. I cannot forget my Master's classmates, Sandra, Ferran, Leire, Nacho and Andrés, who have always been ready to discuss any idea, no matter the time and no matter the day of the week. My gratitude goes very especially to Ferran Martínez and Nacho Lago. I am also very grateful to Alberto Penadés. His work on electoral formulae and his help have been very encouraging. Pepe Fernandez, Braulio Gómez and Sonia Alonso are also part of this dissertation. Pepe has offered his sharp mind to discuss the trickiest points of the dissertation and his observations have been very valuable. Braulio and Sonia have suffered intensely the harder part of my creative process without complaining and with sincere and continuous support. All the hours they have spent listening to me will never be forgotten. Magdalena Nebreda and Jacqueline de la Fuente have made my life easier over the years with respect to all administrative issues. Ivan Blanco and Gonzalo Rivero have spent unvaluable time with the editing of this dissertation, to them my most *sincere gratitude. I must also thank Toñi Ruiz and Susana Aguilar for all their help at the Universidad Complutense de Madrid. The list of people could continue for many pages because these names represent just the tip of a large iceberg. No one in the Center who has spent some time with me should feel excluded from this list, on the contrary, I thank all of them for sharing and exchanging with me bright ideas and good moments.*

Without the financial suppport of the Fundación Cajamadrid and the Fundación la Caixa it would have been impossible to spend the most critical phase of my research at New York University. I have to specially thank these two institutions for having trusted in me and given me a scholarship to travel and study at this university. It would be no exaggeration to say that without their financial support, this dissertation would never have been completed. The time that I spent in New York working very closely with my supervisor was essential and decisive for my research. Life in New York was made easier by Nora Burke who kindly helped me at all times. I also benefited during my stay at the NYU from the altruistic generosity of Matt Golder.

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I thank Adam for having being so extremely patient with me and for having put so much trust in me.

My dissertation has also benefited from the love and support of my friends and family. In the first place, I must thank my friend Ivan not only for his unconditional and true friendship over so many years but also for all the time he spent with me designing the computer programme used in this dissertation. I would also like to thank David Mayor, Javier Valls and Daniel Sarmiento for their constant optimism. Maxi, Güido and Maricruz, my brothers and sister, have always been faithful supporters of the project, and have always given me their sincerest encouragement and understanding, above all in the hardest moments.

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Chapter 1

Introduction

This dissertation is devoted to the analysis of electoral systems. It lies half way between the fields of electoral engineering and comparative politics. It can be embedded within typical electoral engineering studies insofar as it focuses on the mechanical functioning of electoral systems. It is also belongs to the field of comparative politics because its final goal is to offer a characterization of all existing democracies between 1945 and 2000. However, this dissertation is above all an instrument for future research. Its main contribution lies in the creation of a parametric measure which can be used to characterize any conceivable electoral system. The measure proposed here establishes the minimum threshold of votes at the national level that any party must cross in order to have a chance of winning a given number of seats in parliament. I label this measure the aggregated threshold function and this is the tool used to characterize any electoral system. The following pages will be devoted to fully explaining how these functions have been obtained, how these functions have been optimized and, finally, how these functions have been applied.

The mechanical effects of electoral systems.

Electoral systems are institutions that transform political preferences into political representation. The mechanical process in which all the elements that make up this institution interact determines the number of seats that each party obtains in the parliament. Distinct electoral systems can be defined on the basis of this mechanical process.

The literature has already identified some measures that accord with this idea. Some of these are based on how proportional electoral systems are (Gallagher 1991, Loosemore and Hanby. 1971, Monroe 1994). Roughly speaking, electoral systems can be classified depending on how far they deviate from perfect proportionality. An electoral system is said to be perfectly proportional if the share of votes obtained by any given party gives it the same share of the seats. To put this more simply, if a party wins 50% of the vote and wins 50% of the seat, then the electoral system that produces such a result is perfectly proportional. All measures based on this idea can be used to describe any electoral system. One possible shortcoming of this approach lies in its dependence on election results. Measures of proportionality can only be applied when the distribution of votes and seats among all the parties is known. Hence electoral systems cannot be characterized *ex ante,* that is to say, before the election takes place. Only when information about the total share of votes and total share of seats that each party has won is known can one characterize the electoral system in terms of the deviation from perfect proportionality.

This mechanical approach can also be examined from the opposite perspective. Since electoral systems transform votes into seats, we may ask about the number of votes required to win a given number of seats. This is also a mechanical process characteristic of each electoral system. This reasoning provides the basis for other types of measure that can also be used to characterize electoral systems. These measures are based on the minimum and the maximum number of votes that are required to win a seat in the parliament (Rokkan 1968, Rae et. al. 1971, Lijphart and Gibberd 1977, Penadés 2000). In this approach, a set of functions is calculated in order to establish the upper and the lower bounds of an interval of votes that each electoral system produces in order to allow a party to enter parliament. The inclusion and exclusion thresholds are at the center of this discussion. The threshold of representation or threshold of inclusion was first calculated by Rokkan (Rokkan 1968). This value shows the best condition under which a party may win a seat in parliament. The other side of the coin is the value calculated for the threshold of exclusion. This function calculates the worst condition for a party to win a seat. This value was calculated by Rae, Loosemore and Hanby (1971) in response to Rokkan's proposition. Both values can be used to characterize an electoral system. Electoral systems can be, for example, classified depending on how high or low their inclusion or exclusion thresholds are. Since electoral systems define the lower and the upper bounds of the interval of votes required to win the first seat in the parliament, they give us an indicator which can be used to test the extent to which electoral systems allow small parties to win representation in parliament.

The inclusion and exclusion thresholds also provide the basis for an empirical measure that is widely used to characterize electoral systems: the effective threshold. This function has mainly been developed by Taagepera and Shugart (1989) and Lijphart (1994). The values obtained for the effective threshold function represent the average value of both the inclusion and the exclusion thresholds. This is considered to provide a close approximation to the real value that each electoral system establishes in order for a party to win a single seat. However, there are also some problems with these effective threshold values. One of them is that, as they appear in the literature, all functions are calculated at district level. These functions do not provide values at the national level; that is, they do not offer a method to aggregate district values. A second problem of these effective threshold values is that they are not general and cannot be applied to all electoral systems. They are calculated by taking into account only the most widely-used electoral formulae, and hence they do not consider all different existing formulae. As they appear in the literature, threshold functions seems to be designed to fit empirical results rather than to reveal the mechanics of every possible electoral system.

Attempts have also been made to calculate threshold functions not just for one seat but for any number of them (Rae et al. 1971; Lijphart and Gibberd 1977; Penadés 2000). This certainly constitutes an interesting way of measuring the functioning of electoral systems. By establishing the interval of votes for each seat, one can calculate the effects that any electoral systems has on the allocation of parliamentary seats. However, the problems this approach poses have already been mentioned. The bulk of the literature that has treated this issue has concentrated on the district level and has only considered the most common electoral formulae. This does not, of course, invalidate this literature, but it would be more desirable to have a measure with fewer restrictions.

In this thesis I hope to overcome all these shortcomings associated with existing measures. For this reason I have sought to develop a set of functions capable of producing national or aggregated values. This set of functions is also general because it can be applied to every possible electoral formula. In fact, each type of electoral formula has its own threshold function. These aggregated threshold functions can also be used to calculate the proportion of votes required to win any number of seats. Finally, this set of functions can be applied before an election takes place. That is to say, one can apply aggregated threshold functions simply on the basis of the elements that define an electoral system, and not only by looking, after an election, at the results generated by a given electoral system. In this sense, aggregated threshold functions can be termed *ex ante* institutional responses. A starting point is required in order to define aggregated threshold functions. This is provided by an existing measure calculated by Penadés (2000). Penadés calculates district threshold functions as tools to calculate the necessary and sufficient share of the vote to win any number of seats given any district magnitude, any electoral formula and any number of parties. This thesis proposes a method to aggregate the values obtained when applying these district threshold functions. My aggregation method is based on a broader definition of electoral systems and a series of assumptions that will be presented in the pages that follow. The result is a new set of functions that can be used to calculate the share of votes required to win any number of seats nationwide. This set of functions provides the basis to characterize any electoral system.

Why is it necessary to characterize electoral systems?

Elections constitute a defining feature of democracies. As Przeworski points out, democracy is a system in which the incumbent sometimes loses elections (Przeworski 1991). In order for a democracy to exist, therefore, periodic elections must take place. This does not mean that any polity that has periodic elections can be considered to be a democracy. The fact that an incumbent may lose an election implies the existence of a political opposition with a chance of winning the elections and that there are established rules setting up mechanisms to guarantee this political plurality. Elections are a necessary but not a sufficient condition for the existence of democracy. Elections are, therefore, instruments through which voters can replace incumbents and express their political options. Through elections voters can punish or reward governmental performance, as well as decide which policies better fit their own preferences (Przeworski, Manin and Stokes 1999; Bingham Powell Jr. 2000; Maravall 2003).

Given that elections are a key defining feature of any democracy, the study of elections is also the study of democracies. This dissertation contributes to this field by providing a characterization of every electoral process in every democracy in the world between 1945 and 2000. The ultimate goal of this thesis is to introduce a parametric value that summarizes the mechanics of every electoral system. If elections are so crucial to democracy, then the existence of such a characterization should help us understand many phenomena associated with the working of democracy. What institutional design is most likely to be adopted when many political parties compete? Under what electoral system are voters best able to punish or reward political performance? Why are different electoral system chosen? These are some of the questions that can be answered with the help of the characterization proposed in this thesis.

The unit of analysis in this study is the electoral system used for parliamentary elections in each democratic country between 1945 and 2000. Broadly speaking, an electoral system is defined as an institution composed of five elements. The first element is the number of seats in the lower chamber or house of the legislature. The second is the number of parties that compete for those seats. Since elections are competitive, I will assume that the number of parties always equals at least two. The third element is the number of districts into which the country is divided. The number of districts must range from one to the total number of seats in the parliament. When the number of districts is one, then all seats are elected in a single district comprising the entire territory. This is the case in Israel, The Netherlands or Moldova. When the number of districts equals the number of seats in the parliament, the country is divided into uninominal districts. Countries like the United Kingdom, the United States or India have used uninominal districts to choose their members of parliament. The fourth element is a vector that shows the distribution of seats in each district. If the number of districts is a value that ranges from one to the total number of seats in parliament, seats can be distributed in any number of districts that fulfil that property. In Latvia the electoral system used to elect the 100 members of the parliament has 5 districts. Twenty members of the parliament are elected in each of these districts. However, in Spain the 350 seats in parliament are distributed among 52 districts, ranging in size from 1 to about 30 seats. All this information is brought together in a vector that shows how many seats are returned by each of the districts into which the country is divided. Finally, an electoral formula is required to transform votes into seats. In this study, an institution with these five elements is called a complete electoral system.

The characterization proposed here is based on two main two ideas. First, complete electoral systems are characterized in terms of the minimum proportion of votes necessary to win one seat in the parliament. The value expresses the same idea as the threshold of inclusion and indicates the proportion of votes that each party must obtain at national level in order to win one seat in the parliament. In substantive terms, this value provides a measure to test the ease with which the institution allows the entry of small parties into parliament. Complete electoral systems with lower aggregated values for obtaining one seat in parliament are more likely to have a larger number of small parties than those complete electoral systems with higher aggregated values to win a single seat.

The second value used here to characterize electoral systems is the minimum proportion of votes necessary to win the majority of seats in parliament. This value represents the threshold that each party must cross in order to win the majority of seats in the parliament. A party obtaining this proportion of votes does not automatically win the majority of seats in the parliament, but rather it meets the minimum condition that must be satisfied in order to be in a position to do so. In other words, a party that does not receive that proportion of votes has no chance of winning the majority of seats in the parliament. In substantive terms, this value indicates the likelihood of the formation of coalition cabinets in parliamentary systems or strong parliamentary majorities in other regimes. Parliamentary systems with complete electoral systems that result in lower aggregated values required to win the majority of seats in the parliament are more likely to have stronger, single-party cabinets than parliamentary systems with complete electoral systems with a higher value. The probability of the existence of coalition cabinets is logically higher in the latter than in the former.

The results presented in this dissertation offer a portrait of the mechanical operation of each complete electoral system used in the period under study. The aggregated approach described in this dissertation constitutes an innovation in the study of electoral systems. As noted above, existing attempts to characterize electoral systems have either been partial and/or focused on the district level. For these reasons, the elaboration of the aggregated threshold functions as defined here as well as their application to the characterization of most of the electoral system around the world between 1945-2000 may be interpreted not only as a step forward in the field of study, but also as offering a new variable that should make it possible to offer more complete answers to some of the questions mentioned above.

Outline of the thesis.

The structure of the thesis itself reveals how, as noted above, it lies somewhere between the fields of electoral engineering and comparative politics. Even though the characterization of electoral systems used between 1945 and 2000 is made on the basis of their aggregated threshold functions, the emphasis of the dissertation lies elsewhere. The first part of the study focuses on the way electoral systems work and how their mechanical functioning can be summarized in aggregated threshold functions. The second part of the thesis, in contrast, pays more attention to how particular electoral systems used in particular elections can be characterized in accordance with these functions. The emphasis in this part of the study is on the description on specific electoral systems and the results they produced in specific elections.

Enriching the field of electoral engineering, Chapters 2 and 3 explain all the features that combine to define aggregated threshold functions. These two chapters are written in a formal and abstract way. Chapter 2 sets out the theoretical framework within which aggregated threshold functions must be located. The chapter begins with a critical discussion of the different measures and approaches used in the existing literature to characterize electoral systems, taking into account their mechanical functioning. After identifying the relevant theoretical and empirical debates, I go on to explain why threshold functions constitute an appropriate measure with which to characterize electoral systems. Aggregated threshold functions are presented here as representing a step beyond the approach taken by Penadés (2000). The notations and definitions that describe aggregated threshold functions constitute the core of this chapter. Finally, the results of data validation are shown to prove the predictive capacity of these functions.

The aggregated threshold functions are optimized in Chapter Three. The approach here is also formal and mathematical. Briefly, given a total number of seats, the optimization process seeks to identify the combination of seats from all districts that results in the minimum proportion of votes required to win that number of seats. The method used to optimize aggregated threshold functions requires the elaboration of a number of theorems. Attention is first paid to optimization in cases when the total number of seats equals 1. Then, I consider the case in which the total number of seats equals half of the seats in the parliament. This optimization is applied to winner-takes-all and closed list proportional representation electoral systems, while I also offer some intuitions about multi-tier and mixed electoral systems. The theorems are developed, tested and illustrated through examples.

Chapter 4 serves as a bridge between the formal and abstract ideas developed in Chapters 2 and 3 and the empirical application of the functions in Chapters 5 to 7. Chapter 4 has two main goals. The first one is to present the database used in the remainder of the thesis. Data will be classified by families of electoral formulae. Since the unit of analysis is the electoral system, a set of criteria is provided through which to differentiate between electoral systems. Roughly speaking, any two electoral systems will be considered as different if they differ in the number of seats in the parliament, if they use a different electoral formulae, or have different tiers of seat allocation, if the number of districts is different and if there has been a non-democratic period between the two consecutive elections. The second goal of this chapter is to present the methodology required to apply aggregated threshold functions. At this point, the chapter will set out the methodology used for cases in which the information is known for all variables. A proxy function will be elaborated for those cases in which the variable containing the distribution of seats among districts is not known.

Chapters 5, 6, and 7 describe the electoral systems that fit the criteria established in Chapter 4. These three chapters share a very similar structure and their goal is to show the values of the aggregated threshold functions that are used to characterize these electoral systems. Electoral systems are characterized in accordance with the type of electoral formula used. Hence, Chapter 5 focuses on winner-takes-all electoral systems using majority or plurality electoral formulae. The chapter briefly describes the main features of all systems that can be classified under this label and explains which of them can be subjected to the application of aggregated threshold functions. Finally, aggregated threshold data are shown for each electoral system. Data is introduced first for one seat, and then for the majority of seats in the parliament.

Chapter 6 describes the complete electoral systems that fit into the list proportional representation category. List proportional representation electoral systems are divided into two types depending on the type of electoral formula used to transform the votes into seats. The chapter adopts a similar structure for both types of electoral formula. Divisor-based and quota-based electoral systems are presented and formally defined. Once defined, these electoral systems are characterized using the data produced by the aggregated threshold functions. As in the case of winner-takes-all electoral systems, the discussion of aggregated threshold data for one seat is followed by the presentation and analysis of data for winning the majority of seats in the parliament.

Chapter 7 moves on to consider multi-tier and mixed-member electoral systems. As in Chapter 6, a parallel structure is followed. Given the particular features of these electoral systems, the methodology used to apply aggregated threshold functions to these electoral systems is slightly different to that used in the other cases. To put this simply, this is because multi-tier and mixed-electoral systems are characterized by having two or more tiers of seat allocation. The difference between them is that whereas in the former the same electoral formula is used in both tiers, in the latter the formula used is different in each tier. This peculiarity complicates the application of aggregated threshold functions, making it necessary to develop an alternative methodology. This methodology is, however, tentative and rests on strong assumptions. So, after defining these electoral systems and specifying which of them can be subjected to the application of aggregated threshold functions, some data is presented. As in previous chapters, data for winning one seat is shown first and followed by data for winning the majority of seats in parliament.

Finally, Chapter 8 sums up the main conclusions that can be drawn from this thesis.

12/ Aggregated Threshold Functions.

Chapter 2

Characterizing Electoral Systems

The debate about the characterization of electoral systems can be approached from at least three perspectives. On the one hand, one can characterize an electoral system in accordance with the way it converts votes into seats. Or to put it in another way, by measuring the degree of distortion between the share of votes and seats produced by each electoral system. The distortion index, DI, is one measure that illustrates this idea. A second approach to the characterization of electoral systems involves measuring the range of votes with which political parties achieve representation in parliament. Discussion about the exclusion threshold, the threshold of representation, the effective threshold or district magnitudes dominates this approach. Finally, there is an innovative and original approach through which to characterize electoral systems. This approach is based on threshold functions and centres on the necessary and sufficient conditions to obtain a determined number of seats given the magnitude of a district, the electoral formula and the number of competing parties. All these approaches are discussed in this chapter. I conclude by proposing a new measure to characterize electoral systems which consists of a reformulation of threshold functions. The chapter ends by testing the predictive capacity of this new measure using actual electoral data from different electoral systems.

2.1 Traditional approaches to characterizing electoral systems.

There is a vast literature devoted to the mechanical effects of electoral systems. In a seminal article Burt L. Monroe offers a detailed discussion of the most important measures of disproportionality (Monroe 1994). The index created by Loosemore and Hanby in 1971 is probably the most inspiring of these measures, since all other indexes are to some extent derived from it. As these authors explain, given a district of size M_d with P_d competing parties,

> "...distortion, DI, is defined as the extent to which the distribution of seats won does not mirror the distribution of votes cast for all parties" (Loosemore and Hanby 1971: 468)

In accordance with this definition, they propose a function to calculate the distortion of an electoral system:

$$DI = \sum_{p=1}^{P_d} \left| V_d^p - S_d^p \right| \tag{2.1}$$

Where V_d^p and S_d^p respectively refer to the share of votes and number of seats won by party p in district d.

However for practical reasons, Loosemore and Hanby preferred an index with 0 as the minimum value and 1 as the maximum[1]. The function that they went on to propose is thus as follows:

[1]Loosemore and Hanby justified this new function through the following example. Imagine that there are two competing parties, P_d= 2, in a single member constituency, M_d= 1. Imagine also that p_1 obtains all the votes but no seats and p_2 wins all the seats but none of the votes cast. In this hypothetical situation the maximum value of DI would be 2. This situation, though in

$$DI_{L\&H} = \frac{1}{2} \sum_{p=1}^{P_d} \left| V_d^p - S_d^p \right| \tag{2.2}$$

According to the literature, one of the problems with this index concerns its over-sensitivity to the number of parties. It is for this reason that Rae (1971), Lijphart (1984) and Gallagher (1991) have proposed modifications to Loosemore and Hanby's original index. However, these new indexes also suffer from their extreme sensitivity to the incorporation of minor parties (in the case of Rae) or loss of information (Lijphart). Gallagher (1991) moves beyond Loosemore and Hanby's approach and explains why the concept of proportionality is relative and how each electoral system has its own index of disproportionality. So, building on largest remainders methods he proposes the 'least square index':

$$DI_{lsi} = \sqrt{\frac{\sum_{p=1}^{P_d} \left(V_d^p - S_d^p \right)^2}{2}} \tag{2.3}$$

Based on the d'Hondt or greater divisors he derived the following function:

$$DI_{d'Hondt} = \max(L) \tag{2.4}$$

Where L is the largest remainder among all parties with a share of the vote,
$V_d^p > 0.5$

Finally, Gallagher offers an index based on the Sainte-Laguë electoral formula:

$$DI_{S-L} = \sum_{p=1}^{P_d} \frac{\left(V_d^p - S_d^p \right)^2}{V_d^p} \tag{2.5}$$

practice impossible, is revealing in theoretical terms in order to understand the limits of this value (Lossemore and Hanby 1971:469).

While it is true that the definition of each electoral system may implicitly contain its own concept of proportionality, the indexes proposed by Gallagher may be taken as adequate measures insofar as, in Monroe's words, they minimize "an infinite number of indexes, any of which might be chosen" (Monroe 1994:140).

The crucial point for the purposes of this research is that all these indexes reflect the mechanical effect of the electoral system once an election has occurred. As noted above, all the measures discussed here show the variation in the ratio between seats and votes once the election results are known. That is, they are measures that explain the mechanical effect of electoral systems *ex post.* This is precisely one reason why these indexes should not be used to characterize electoral systems. If one really wants to see how each electoral systems operates, then it should be possible to know its outcomes *ex ante,* that is, before elections are held. Only in this way would it be possible to anticipate the electoral outcomes of each system. In a bid to overcome this limitation, one of the objectives of this thesis is to develop a measure that takes into account the defining variables of an electoral system and is able to identify their mechanical effects without relying on electoral results.

One way of approaching this task is by considering the degree of openness of each electoral system. As Taagepera points out, one of the most distinctive features of electoral systems are the constraints that are established to the entry into parliament of small political parties (Taagepera 1998a). These constraints have a direct influence on the number of political parties that gain representation and also reflect the deviation from perfect proportional representation. When talking about electoral constraints, legal threshold, T_L, and district magnitude, M_d must be taken into account.

Mainly based on district magnitude, there are two important measures that clearly illustrate the degree of openness of an electoral system. As I will explain in more details in the next section, the threshold of inclusion or threshold of representation refers to the percentage of votes below which a party will not win a seat. In other words, this threshold points to the minimum percentage of votes which a party needs, in the most favorable circumstances, to obtain a seat in a district, given the district magnitude (Lijphart 1994, Grofman 2001). A share of votes higher than the threshold of inclusion is a necessary (though not sufficient) condition to obtain representation (Penadés 2000).

In Lijphart (1994) this threshold is calculated using the following function:

$$T_{inc} = \frac{100\%}{2M_d} \tag{2.6}$$

Where M_d, as already noted, indicates the magnitude of district d.

The threshold of exclusion is the maximum percentage of votes which, under the most unfavorable circumstances, may be insufficient for a party to win a seat given the district magnitude. Surpassing this threshold is a sufficient (though not necessary) condition to win a seat. Or to put it another way, a party that gets a percentage of votes above this threshold will win at least one seat (Lijphart 1994, Penadés 2000).

Again, Lijphart (1994) calculates this threshold using the following function:

$$T_{exc} = \frac{100\%}{M_d + 1} \tag{2.7}$$

These two thresholds constitute the basis for the most commonly used measure to characterize electoral systems. The effective threshold, T^{eff}, is a value that lies between the inclusion and the exclusion threshold. Or as Carles Boix explains, the notion of the effective threshold is

> "...based on the idea that the percentage of votes a party needs to gain representation is not a specific number but a range of possibilities" (Boix 1999:614; see also Lijphart 1994:25).

In fact, Lijphart takes the average of the exclusion and inclusion thresholds in order to calculate the effective threshold. This gives us the following function:

$$T_{Lijp}^{eff} = \frac{50\%}{M_d + 1} + \frac{50\%}{2M_d}, \ for \ M_d > 1 \tag{2.8}$$

This expression can be contrasted with the function proposed by Taagepera and Shugart (1989):

$$T_{T\&S}^{eff} = \left[\frac{1}{M_d + 1} + \frac{1}{M_d P_d} \right] /2 \tag{2.9}$$

Where P_d is the number of competing parties in district d and where it is assumed that P_d is about the same size as district magnitude, M_d, except for $M_d = 1$ in which case, P_d must be at least equal to 2 (Taagepera and Shugart 1989:274-277). Taking this into account, Taagepera and Shugart's function is reduced to

$$T_{T\&S}^{eff} = \frac{50\%}{M_d} \tag{2.10}$$

Discrepancies arose between Lijphart and Taagepera about the validity of their formulation for the case of $M_d = 1$. When $M_d = 1$, both Taagepera´s and Lijphart´s functions yield a result of 50% which is too high, given that according to their data single-member districts seats are won with fewer votes. Therefore, Lijphart opted arbitrarily for a threshold of 35% but Taagepera proposed in a private communication to Lijphart (Lijphart 1994) a new function:

$$T_{Taag}^{eff} = \frac{75\%}{M_d + 1} \tag{2.11}$$

Using this function, when $M_d = 1$, $T_{Taag}^{eff} = 37.5\%$, which is very much in accordance with Lijphart´s recommendation of 35%.

The utility of the inclusion and exclusion thresholds is straightforward since they make it possible to calculate the range of votes within which political parties will win their first seat. However, I argue that there are some problems in this respect which are worth noting. The first is that these thresholds are not valid for all electoral systems. As I demonstrate below, each electoral formula has its own inclusion and exclusion thresholds (Penadés 2001; Lijphart and Gibberd 1977). This problem becomes even more acute when we consider how the effective threshold is calculated. Taagepera and Shugart (1989) used, for example, the divisor-based d'Hondt electoral formula to calculate the exclusion threshold and the quota-based Hare electoral formula to calculate the inclusion threshold. And while agreeing with this method of calculating the exclusion threshold, Lijphart disagreed with them with respect to the appropriate method of calculating the inclusion threshold. Instead of using a largest remainder method with full Hare Quota, he proposed using one half of it, $\frac{1}{2M_d}$. While it is true that all these measurements served Lijphart's and Taagepera and Shugart's purposes well, they provide no mechanisms or logical procedures to explain why this is so. They are empirical but not systematic measures. Hence, the need for a unifying and more universal function. This becomes even more apparent when it is remembered that the studies reviewed here refer mainly to the district-, as opposed to the national or aggregated-level [2].

There is, finally, another interesting approach to characterizing electoral systems. Penadés (2000) defines the threshold functions of an electoral system. This approach is built around the electoral for-

[2] In this respect, an illuminating discussion about nationwide thresholds of exclusion and inclusion can be found in two interesting articles by Taagepera (Taagepera 1998c, 2002). However, these articles suffer from some of the same problems mentioned here. Taagepera does provide a formula to calculate both the exclusion and inclusion thresholds (Taagepera 1998c). However, he only focuses on a formulation based on the d'Hondt allocation rule simply because his reasoning is only valid for this specific formula. Furthermore, his reasoning is not valid, for example, for systems such as the German one which has compensatory seats, a formula which has also been adopted in some Central and Eastern European countries (for example Albania). These articles do, however, clarify some issues, and hence will be taken into consideration in this research. This is the case for example, of his discussion of the effective number of competing-parties or the number of electoral districts (Taagepera 1998c, 2002)

mula used. Threshold functions determine at the district level the necessary and sufficient number of votes for each seat as a function of the district magnitude, the electoral formula and the number of competing parties, whatever the distribution of votes among them (Penadés 2000:33). Penadés´s work is theoretical and needs further development. However, the path he has opened is promising and appealing since it points to the possibilities for the rigorous development of a general formula that can be applied to all systems. I explain this approach in detail in the following section.

2.2 Threshold functions: first steps towards an alternative measurement to characterize electoral systems.

Stein Rokkan (1968) posited a key question in the study of electoral systems:

> How little support can possibly earn a party its first parliamentary seat? (Rokkan 1968:6-21).

This question is sparked by his keen interest in identifying the characteristic range of votes required by each electoral system for a party to win a seat. In other words, the issue at stake is about calculating the best and worst conditions to win parliamentary representation under any electoral system. The pioneering work in this area is Rokkan's calculation of the most favorable condition to win one seat. The question that Rokkan tries to answer is how little support a party needs in order to win its first seat in parliament. This concept is known as the threshold of representation and refers to the minimum share of votes that allows any party, p, to win 1 seat in a district. Rokkan (1968) calculates this condition for three types of electoral formulae: d'Hondt, Sainte-Laguë and Hare.

The concept of threshold of representation was contested by Rae et al (1971). Rather than focusing on the most favorable condition that allows a party to win a seat, Rae, Hanby and Loosemore were more concerned with the maximum support a party could win and still not obtain a seat. This is the threshold of exclusion, which introduces the idea that a party may not win a seat even though it has strong political support. The threshold of exclusion was calculated for the same three electoral formulae used by Rokkan, as well as for the plurality formula, all applied at district level.

These theoretical developments still leave one question unanswered. One could ask if such thresholds might be calculated not just for one seat but for any number of them in a multi-member district. Rae et al. proposed some possible answers to this question in their work (1971:485). However, the most rigorous attempt to find such a threshold was Lijphart and Gibberd (1977). Here, Lijphart and Gibberd refined the threshold of inclusion or exclusion calculated by Rokkan and Rae et al. and expanded the analysis to a new divisor-based electoral formula: Modified Sainte-Laguë. But their most valuable contribution was to develop formal reasoning to calculate these values for any number of seats. This is what they term "payoffs functions" (Lijphart and Gibberd 1977: 230).

These studies are of great value. However, one may still ask if it is possible to identify some functions capable of calculating these thresholds for any electoral system using any electoral formula. As I pointed out, Lijphart and Gibberd's study only calculated the values for the most commonly used electoral formula: d'Hondt, Sainte-Laguë, Modified Sainte-Laguë and Hare. However, the universe of electoral formulae is much more broader, raising the question of whether it is possible to identify a function capable of calculating the best and the worst conditions to win any number of seats in a district when any electoral formula is applied. In other words, can we define a general function that could be applied to any electoral system using any conceivable electoral formula. One excellent and extremely attractive approach to this problem lies in what Penadés (2000) calls threshold functions.

Before discussing this, it is perhaps necessary to refer to the terminology that will be used from this point on. As noted above, Lijphart and Gibberd used the term "payoff functions" to refer to the share of the vote needed to win any number of seats in a district. However, as Penadés rightly argues, "payoff functions" should be used to refer to those functions that predict the number of seats that a party wins given its share of votes, whereas threshold functions should refer to a set of functions used to calculate the minimum and maximum proportion of votes needed to win a determined number of seats in a district (Penadés, 2000:35). The term threshold functions is used here in this second sense.

Penadés (2000) theoretically refines and enriches the work of the other scholars mentioned above by calculating a general and universal formulation for threshold functions. His study proposes threshold functions for both majoritarian and proportional representation electoral systems. Among the latter, he distinguishes between divisor-based and quota-based electoral systems. These threshold functions apply, then, for any number of seats and for any possible electoral formula. As in Lijphart and Gibberd, Penadés' threshold functions make it possible to calculate the necessary and sufficient number of votes that a party must obtain in order to win a determined number of seats given any elemental electoral system. An elemental electoral system comprises three components: an electoral formula, F, a district magnitude, M_d, and finally the number of competing parties in that district, P_d(Penadés, 2000:23).

Threshold functions are calculated by Penadés at district level and on the basis of two specific elements of the electoral formula: the modifier of the quota, n, for quota-based electoral formulae[3]

[3]Briefly, a quota, $Q(n)$, is defined as

$$Q(n) = \frac{V_d}{M_d + n}$$

where M_d is the magnitude of district d, n is the modifier of the quota and V_d corresponds to the total of valid votes in district d. When $n = 0$, the Hare quota or simple quota is obtained; when $n = 1$, the Droop quota and when $n = 2$, the Imperiali quota is obtained. See Chapter 6 for a much more detailed account of how quota-based electoral systems are conceived.

and the adjustment term, c, for divisor-based electoral formulae[4]. These two concepts derive from a redefinition of the working of these two types of formulae. Given the existence of these two families of electoral formulae, two approaches will also be used to calculate threshold functions. The first is based on n, the modifier of the quota, and the second on the adjustment term, c.

Before I introduce the general form of these formulae, it would be useful to see how threshold functions are conceived. One way of approaching these functions is by working out the necessary and the sufficient number of votes to obtain the same number of seats (Rae et al.1971; Lijphart and Gibberd 1977; Penadés 2000). It should be remembered that the threshold of representation refers to the share of the vote below which it is impossible to obtain representation. In this sense, the minimum number of votes required to win one seat is also the necessary number of votes to obtain representation. If a party, p, wins a share of the vote which equals the necessary number of votes to obtain a given number of seats in district d, S_d^p, that party will have a chance of winning that amount of seats in that district. Likewise, the threshold of exclusion is the percentage of votes at which it is impossible for a party not to obtain representation. That is to say, the sufficient number of votes to win one seat equals the maximum number of votes with which a party will obtain no seats. If a party, p, obtains a share of the votes which is above the sufficient number of votes required to win S_d^p seats, that party will get at least that amount of seats in district, d.

This reasoning gives us two important definitions that should make it possible to fully understand threshold functions.

Definition 1 $V_{S_d}^{nec}(F, M_d, P_d)$: *This function determines the necessary, but not sufficient, share of votes in order to obtain any number of seats in district* d, S_d, *provided that* S_d *is a number smaller*

[4]Very briefly, the adjustment term, c, is used to allocate seats in divisor-based electoral formulae. When $c = 1$, then d'Hondt electoral formula is used, when $c = 0.5$, the Sainte-Laguë electoral formula is used. See Chapter 6 for a much more detailed exploration of divisor-based electoral formulae.

or equal to district magnitude, $1 \leq S_d \leq M_d$ *and given the electoral formula,* F*, the magnitude of district d,* M_d*, and the number of competing parties in that district,* P_d*.*

Definition 2 $V^{suf}_{S_d}(F, M_d, P_d)$*: This function determines the sufficient number of votes to obtain* S_d *seats given* F, M_d *and* P_d *and provided that* $1 \leq S_d \leq M_d$ *(Penadés 2000:127).*

From these definitions, we derive the following conclusions for any party, p, with a share of votes, V^p_d, given F, M_d and P_d :

Conclusion 1 *If* $V^p_d \leq V^{nec}_{S_d}(F, M_d, P_d)$*, then* $S^p_d \leq S_d$*.*

Conclusion 2 *If* $V^p_d \geq V^{suf}_{S_d}(F, M_d, P_d)$*, then* $S^p_d \geq S_d$*.*

Conclusion 3 *If* $V^{nec}_{S_d}(F, M_d, P_d) \leq V^p_d \leq V^{suf}_{S_d}(F, M_d, P_d)$*, then* $\max(S^p_d) = S_d$ *provided*[5] *that* $V^{nec}_{S_d+1} \geq V^{suf}_{S_d}$.

Threshold functions comprise, therefore, two different functions. On the one hand, the function of necessary votes for any number of seats, $V^{nec}_{S_d}(F, M_d, P_d)$, and on the other, the function of sufficient votes for those S_d seats, $V^{suf}_{S_d}(F, M_d, P_d)$.

Once these definitions have been established, the threshold functions for both, quota-based and divisor-based methods are as follows:

- Quota-based methods

$$V^{nec}_{S_d}(M_d, P_d, n) = \frac{P_d(S_d - 1) + 1 + n}{P_d(M_d + n)} \tag{2.12}$$

where $1 \leq S_d \leq M_d$ and $n > -M_d$

[5] This restriction is important. If $V^{nec}_{S_d+1} \leq V^{suf}_{S_d}$, then the necessary condition to take S_d+1 seats would lie within the range $\left[V^{nec}_{S_d+1}, V^{suf}_{S_d}\right]$ and conclusion 3 would no longer hold.

$$V_{S_d}^{suf}(M_d, P_d, n) = \begin{cases} \dfrac{P_d(S_d-1)+P_d-1+n}{P_d(M_d+n)} & \text{if } S_d \leq M_d - P_d + 2 \\ \dfrac{S_d-1}{M_d+n} + \dfrac{M_d-S_d+1+n}{(M_d-S_d+2)(M_d+n)} & \text{if } S_d \geq M_d - P_d + 2 \end{cases} \tag{2.13}$$

where 0$\leq S_d \leq M_d - 1$

Threshold functions enable us to establish a general formula for the inclusion and exclusion threshold:

$$V_{S_d=1}^{nec}(M_d, P_d, n) = \frac{n+1}{P_d(M_d+n)} \tag{2.14}$$

If $V_d^p < V_{S_d=1}^{nec}(M_d, P_d, n)$, then it is impossible for party p to obtain a seat in parliament.

$$V_{S_d=1}^{suf}(M_d, P_d, n) = \begin{cases} \dfrac{P_d-1+n}{P_d(M_d+n)} & \text{if } P_d \leq M_d + 1 \\ \dfrac{1}{(M_d+1)} & \text{if } P_d \geq M_d + 1 \end{cases} \tag{2.15}$$

If $V_d^p > V_{S_d=1}^{suf}(M_d, P_d, n)$, then a party p will necessarily obtain at least one seat in parliament.

- Divisor-based methods.

$$V_{S_d}^{nec}(M_d, P_d, c) = \frac{S_d-1+c}{M_d-1+P_dc} \text{ if } c \geq 0 \text{ and } 1 \leq S_d \leq M_d \tag{2.16}$$

$$V^{suf}_{S_d}(M, P_d, c) = \begin{cases} \dfrac{S_d - 1 + c}{M_d + 1 + P_d(c-1)} & \text{if } S_d \leq M_d - P_d + 2 \\ & \text{and } 0 \leq c \leq 1 \\ \dfrac{S_d - 1 + c}{(M_d + 1)c + S_d(1-c) - 1 + c} & \text{if } S_d \geq M_d - P_d + 2 \\ & \text{and } 0 \leq c \leq 1 \\ \dfrac{S_d - 1 + c}{M_d - 1 + 2c} & \text{if } c \geq 1 \end{cases} \tag{2.17}$$

where $0 \leq S_d \leq M_d - 1$

Finally, the general form of both the exclusion and the inclusion thresholds for divisor-based methods are:

$$V^{nec}_{S_d=1}(M_d, P_d, c) = \frac{c}{M_d - 1 + P_d c} \text{ if } c \geq 0 \tag{2.18}$$

and

$$V^{suf}_{S_d=1}(M_d, P_d, c) = \begin{cases} \dfrac{c}{M_d + 1 + P_d(c-1)} & \text{if } P_d \leq M_d + 1 \text{ and } 0 \leq c \leq 1 \\ \dfrac{1}{M_d + 1} & \text{if } P_d \geq M_d + 1 \text{ and } 0 \leq c \leq 1 \end{cases} \tag{2.19}$$

At this point, it is necessary to make a number of observations about the importance of the legal threshold, T_L. Some electoral laws establish a minimum percentage of votes required to win parliamentary representation. These types of electoral barrier, which are largely intended to prevent the entry of minority parties into parliament, are known as legal thresholds. This variable is, of course, of great importance when considering the necessary and sufficient conditions to obtain S_d seats. If $T_L > V^{suf}_{S_d} > V^{nec}_{S_d}$ then, T_L gives us the share of votes that are required to win S_d seats. I will return to the question of the importance of legal thresholds in Chapter 4.

Penadés' contribution to this field derives not only from his success in providing generally applicable inclusion and exclusion thresholds, but also in presenting a function that is capable of calculating the necessary and sufficient number of votes required to win any number of seats. Another important point must be noted in this respect: these functions are conceived for cases in which there is uncertainty about the distribution of votes between all political parties. If we know how the votes are distributed between all competing parties, that is, after the elections have taken place and once the results have been announced, it is possible to make a straightforward calculation of the number of seats that each party will get. Threshold functions indicate the range of votes within which a determined number of seats can be won. The exact distribution of seats can only be calculated once we know the distribution of votes among the competing parties. This is why threshold functions make it possible to make predictions *ex ante* the elections take place. The next two examples provide a detailed example of this idea.

Example 1 *The relationship between threshold functions and the general algorithm for a divisor-based electoral formula (d'Hondt)*

Imagine an electoral system where the size of the district, M_d, is 3, the number of competing parties for those seats is $P_d = 5$ and the electoral formula used to distribute the seats is d'Hondt, $c = 1$. Now, imagine the following distribution of seats:

Using the general algorithm[6] to work out the number of seats that each party would get under the d'Hondt electoral formula, party A would obtain 2 seats and party B the third seat. The rest of the parties would not win any seats. In Table 2.1 above, we can

[6] An algorithm is set of operations related to a calculation. In the context of electoral formulae, algorithms define the procedures used to allocate seats. For example, in Sainte-Lagüe divisor-based formula, the number of votes for each party is divided by$\{0.5, 1.5, 2.5...\}$. In the case of the D'Hondt electoral formula, the divisors are $\{1, 2, 3..M\}$. Seats are allocated to those parties that obtain the highest divisors.

Table 2.1: Allocation of seats using d'Hondt general algorithm

Parties	**Votes** (V_d^p)	V_d^p**/1**	V_d^p**/2**	V_d^p**/3**	**Seats allocated**
A	410	**410**	**205**	137	**2**
B	350	**350**	175	117	**1**
C	190	190	95	63	0
D	30	30	15	10	0
E	20	20	10	6.7	0

see how seats are allocated in accordance with the criterion of the highest average. Whereas this example shows the exact allocation of seats given the number of votes won by each party, threshold functions give us a particular range of votes for each of the seats at stake. The values of the necessary number of votes and the sufficient number of votes for each seat, given M_d, P_d and c are shown below:

Table 2.2: Threshold values for c=1.

Seats (S_d)	$\mathbf{V}_{S_d}^{nec}$	$\mathbf{V}_{S_d}^{suf}$
1	0.14	0.25
2	0.28	0.50
3	0.42	0.75

Table 2.2 shows that in order to have a chance of winning 1 seat, a party must obtain at least 14% of the votes. The party in question will, without any doubt, win its first seat once the proportion of votes obtained by the party is 25% or more. Since party A and B obtain 41% and 35% of the votes respectively, each is entitled to at least 1 seat. However, they are both in a position to win a second seat because the necessary condition to obtain this is 28% or more of the vote. However, since none of the parties has won more than 50% of the vote, the second seat is not guaranteed. In this example, the remaining seat goes to party A since 41% of the votes is higher than 35%, and a property of the electoral formulae is pos-

itive monotony, i.e. they cannot allocate more seats to a party with fewer votes (Penadés 2000:25). Notice how party C with 19% of the vote does not obtain representation. The share of votes obtained by C certainly comes between the range required to obtain the first seat $[0.14, 0.25]$, but since it does not surpass the sufficient number of votes required to obtain this seat, that seat goes to party A.

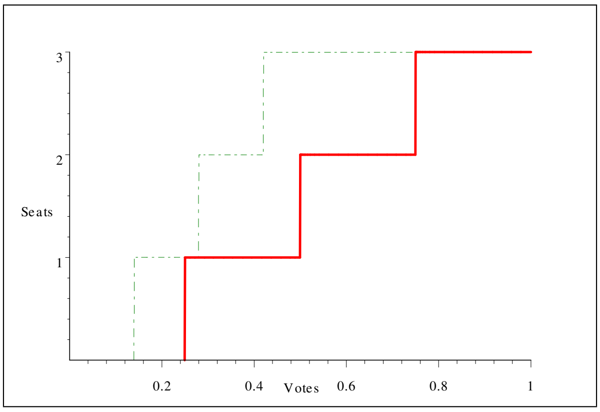


Figure 2.1: Threshold results for $M_d = 3$, $P_d = 5$ and $c = 1$

Figure 2.1 shows the points at which a determined number of seats can be gained. The dot-dash line shows the maximum number of seats that it is possible to win; the solid line shows the number of seats that will certainly be won. We can see from this figure that the first seat will necessarily be won by a party taking 25% of the vote and the third by winning 75% of the vote. If a party wins less than 14% of the vote it is impossible for it to obtain representation and between 25% and 42% the minimum number of seats that can be obtained is 1 because the sufficient condition to obtain a second seat is to take more than 50% of the vote. However a share of the vote of above 42% gives the party in question a chance of obtaining a maximum of 3 seats, whereas a share of between 28% and 42% only gives a party a chance of winning 2 seats maximum. This figure clearly illustrates the idea behind threshold functions. These functions do not tell us how many seats a party will obtain given its electoral result, but rather the number of votes that are required to win a determined number of seats: for party A to gain 2 seats for sure, it must obtain at least 50% of the votes given M_d, P_d and c.

Example 2 *The relationship between threshold functions and the general algorithm for a quota-based electoral formula (Hare).*

Imagine a new electoral system in which the size of M_d is also 3, the number of competing parties for those seats is, once again, $P_d = 5$, but the electoral formula used to distribute the seats is now a Hare quota-based electoral formula ($n = 0$), where the quota, $Q(n) = 333.33$. Now, imagine the following distribution of seats:

Table 2.3: Allocation of seats using the Hare general algorithm

Parties	**Votes** (V_d^p)	$\frac{V_d^p}{Q(n)}$	Z_p	r_p	**Seats allocated**
A	410	**1.23**	**1**	0.23	**1**
B	350	**1.05**	**1**	0.05	**1**
C	190	**0.57**	0	**0.57**	**1**
D	30	0.09	0	0.09	0
E	20	0.06	0	0.06	0
Total	1000	3	2	1	3

Table 2.3 shows the distribution of seats for an electoral system with a quota-based electoral formula, the Hare formula. Given the distribution of the vote, and the values of the quota and largest remainder, each party receives one seat. The results for the threshold function for this electoral system are shown in Table 2.4.

In accordance with the general algorithm, i.e. the rule for allocating seats for a given distribution of the vote, it can be seen that parties A, B and C win one seat each. The results obtained from the threshold function also anticipate this allocation. Parties A and B both take more than 25% of the votes and therefore they win at

Table 2.4: Threshold values for n=0.

Seats (S_d)	$\mathbf{V}_{S_d}^{nec}$	$\mathbf{V}_{S_d}^{suf}$
1	0.06	0.25
2	0.40	0.55
3	0.73	0.83

least one seat each. Party B has won 35% of the vote, with the result that it is impossible for it to obtain 2 seats. The remaining seat is disputed between parties A and C because the former won 19% of the vote, which gives it a chance of winning a seat, as in fact would happen in this case. Threshold functions do not allocate seats but refer to the conditions under which each seat can be obtained. Figure 2.2 illustrates this idea more clearly.

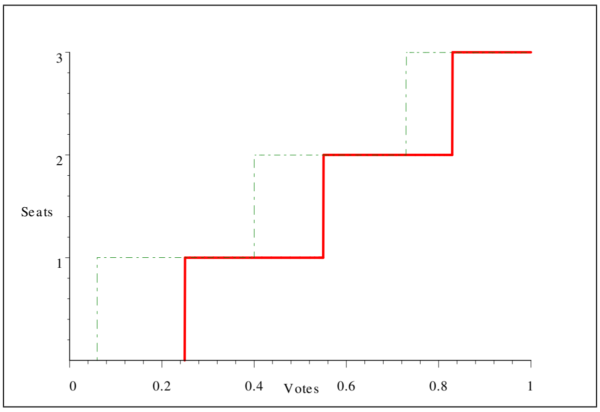


Figure 2.2: Threshold results for $M_d = 3$, $P_d = 5$ and $n = 0$

As noted above, these functions are calculated at district level. An aggregate form of these functions is required for the purposes of this research. In the next section I propose such functions.

2.3 Aggregated Threshold Functions

Seats are allocated according to the number of votes that each party wins. In other words, the number of seats won by a party in a district, S_d^p, is a function of the number of votes it obtains, V_d^p given the other parties' share of the vote. Formally,

$$f : V_d^p \rightarrow S_d^p \tag{2.20}$$

We can obtain one possible measure that could be used to characterize electoral systems if we are able to identify the following:

a) The minimum necessary value of V_T^p in order that that $S_T^p = \frac{M}{2}$

b) The minimum necessary value of V_T^p in order that $S_T^p = 1$

where V_T^p refers to the total share of votes won by party p in all the districts, S_T^p refers to the sum of all seats won in each district by party p and M refers to the size of the parliament.

Once we know these two values, it will be possible to predict, first, how likely an electoral system is to promote coalition governments and second, the level of atomization in the parliament. If the value of V_T^p in case a) is too high then, the likelihood of having a divided government is greater than with a lower value of V_T^p. The value of V_T^p in case b) indicates the degree of openness of the electoral system: if V_T^p is too low then we would expect a much more fragmented parliament than in the case of a higher value of V_T^p.

The value of V_T^p in case b) shows the idea embodied in the threshold of inclusion. It will be remembered from the first section that this function tells us the percentage of the vote, required to obtain the minimum representation. As I also noted above, the literature shows that each electoral formula has a different threshold of inclusion (Lijphart and Gibberd 1977, Gallagher 1991, Penadés 2000). It would be desirable, then, to have a unified method to calculate the value of V_T^p so that S_T^p equals 1 as well as to calculate the value of V_T^p at which S_T^p equals $\frac{M}{2}$.

Aggregation means the addition of individual values. As shown in the previous section, threshold functions are defined at district level. In order to characterize an electoral system on the basis of the criteria defined above it would be useful to have a global measure capable of telling us the best and worst conditions in which a party will win a given distribution of seats among all districts. Hence a method of aggregation is necessary in order to identify this measure. The following notation shows the essentials of this method.

Notation 1 *Aggregated threshold functions will be based on both* $V_{S_d}^{nec}(F, M_d, P_d)$ *and* $V_{S_d}^{suf}(F, M_d, P_d)$ *as defined in the previous section. The reason for proceeding in this way is that the purpose of these functions is to make it possible to calculate the intervals of votes that will define the necessary and sufficient conditions for any party to win* S_T^p *seats at the aggregate level.*

Notation 2 *The number of districts, D, is a parameter that must be taken into account. The number of districts refers to all the constituencies into which the territory is divided. This number ranges from 1 - the whole country is a unique constituency (e.g.Moldova)- to the size of the assembly - single member constituencies (e.g. United Kingdom).*

Notation 3 *Since electoral territories are divided into districts, a vector of size* $1xD$ *can be established.* $\mathbf{M}_d$*, is defined as the vector that contains all district magnitudes in the territory,* $\mathbf{M}_d = [M_1, ..., M_D]$*, where* M_d *refers to the size of district* d*. Also note that* $\sum_{d=1}^{D} M_d = M$*, where* M *refers to the total number of seats in the parliament. Whether the territory has a single district, single-member districts, or more than one tier, seats are won at district* *level*[7]*. Hence, we can define another* $1xD$ *vector,* $\mathbf{S}_j$*, this indicating a particular distribution of seats won in each district. Formally, this vector is expressed as* $\mathbf{S}_j = [S_{dj}, ..., S_{Dj}]$*, where* S_{dj} *indicates the number of seats won in district* d *in a particular distribution of seats,* j*. Since aggregation is understood as the sum of individual values, the aggregated value of seats,* S_T*, is the sum of all the elements that form vector* $\mathbf{S}_j$*. So,*

$$S_T = \sum_{d=1}^{D} S_{d_j} \tag{2.21}$$

provided that

$$0 \leq S_{d_j} \leq M_d \tag{2.22}$$

and

$$0 \leq S_T \leq M \tag{2.23}$$

Notation 4 *It should also be noted that two or more particular combinations of seats,* j*,* $k...z$*, may have the following property:*

$$\sum_{d=1}^{D} S_{d_j} = \sum_{d=1}^{D} S_{d_k} = ... = \sum_{d=1}^{D} S_{d_z} = S_T \tag{2.24}$$

In other words, different combinations of seats may produce the same aggregated number of seats, S_T *.*

Notation 5 *The total share of votes won by party p,* V_T^p*, can also be disaggregated in a* $1xD$ *vector compounded by all individual shares of the vote won in each district. Thus,* $\mathbf{V}^p = \left[V_d^p, ..., V_D^p\right]$ *, where* V_d^p *refers to the share of the vote won in district d by party p and*

$$V_T^p = \sum_{d=1}^{D} V_d^p \tag{2.25}$$

[7] In mixed electoral systems, for example of the type used in countries that choose half of their parliament from single-member districts and half from a single district, both tiers allocate seats at district level, one district being uninominal, the other one having the size $\frac{M}{2}$.

Notation 6 *In the light of Notations* ***3****, 4 and 5, electoral formulae are functions that allocate seats according to the share of the vote won by each party, thus,* $\forall\ \mathbf{V}^p\ \exists\ \mathbf{S}_j^p$. *So, for every particular distribution of votes among all districts that each party obtains,* $\mathbf{V}^p$, *there is one and only one particular distribution of seats,* $\mathbf{S}_j^p$ *which produces an aggregated number of seats,* S_T^p.

Notation 7 *Given that districts may have different magnitudes, a measure to weigh each district must be incorporated into the function. A district encompassing more than* 50% *of the electorate must have a greater weight in the function that a district encompassing just* 10% *of the voters. This measure is based upon the size of the parliament,* M, *and has the following form:*

$$Weight = \frac{M_d}{M} \tag{2.26}$$

Notation 8 *All parties are distributed on a* $1xP$ *vector, so that* $\mathbf{P} = \left[\begin{array}{cccc} 1, & 2, & ..., & P \end{array}\right]$ *where* P *is the total number of parties competing in the whole territory. Since a distinctive feature of any democracy is electoral competition, then* $P \geq 2$.

Condition 1 *Finally, in this research it is assumed that all district magnitudes are commensurate with their voting population. In other words, magnitudes are designed in line with a ratio between voters and seat*[8]. *Hence this research does not consider the possibility of any possible unequal distribution of seats to different territories.*

Definition 3 $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$: *Given a particular distribution of seats* $\mathbf{S}_j$ *won in a distribution of districts,* $\mathbf{M}_d$, *the size of the parliament, the electoral formula and the number of competing parties,* $V_{S_T}^{nec}$ *defines the minimum number of votes that a party needs to win* S_T *seats distributed according to* $\mathbf{S}_j$. *So, if a party*

[8] Bulgaria, Poland or Slovakia are countries where the criterion to distribute seats according to the population is used.

expects to win those S_T seats distributed according to $\mathbf{S}_j$, its total share of the vote must be at least equal to $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$.

Definition 4 $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$*: Given a particular distribution of seats $\mathbf{S}_j$, $V_{S_T}^{suf}$ defines the sufficient condition for a party to obtain S_T seats distributed according to S_j in an electoral system with a parliament of size M, a distribution of districts,$\mathbf{M}_d$, an electoral formula F and a number of competing parties, P. To be sure of wining S_T seats distributed according to S_j, a party must obtain a share of the vote higher than $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$.*

From definitions **3**, and **4** the following conclusions can be inferred:

Conclusion 4 *If $V_T^p < V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$, then $S_T^p < S_T$.*

Note from Notation **4**, however, that two or more particular distributions of seats, $\mathbf{S}_j$, $\mathbf{S}_k ... \mathbf{S}_z$, may produce the same total number of seats, S_T. So if $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, F, P) < V_T^p$, then party p will fulfill the minimum condition to win S_T seats.

Conclusion 5 *If $V_T^p > V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$, then $S_T^p \geq S_T$*

Conclusion 6 *If $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) \leq V_T^p \leq V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$, then* $\max(S_T^p) = S_T$

Mathematically, the aggregated threshold functions can be expressed as:

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^{D} \frac{M_d}{M} \left(V_{S_d}^{nec} \right) \tag{2.27}$$

and

$$V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^{D} \frac{M_d}{M} \min \left\{ \begin{array}{l} V_{S_d}^{suf} \\ V_{S_d+1}^{nec} \end{array} \right. \tag{2.28}$$

To see how this method works, imagine the following situation. Country X has a population of 1000 voters and an assembly of 10 members ($M = 10$) which are unequally distributed in three districts ($D = 3$). The unequal distribution is due to the uneven spread of the population in the country. Each seat has 100 voters and the first district has a population of 500 voters, hence 5 members of the parliament are elected here ($M_1 = 5$). The second district has a population of 300 so $M_2 = 3$ and the third district the remaining 200 ($M_3 = 2$). Country X also has five parties ($P = 5$) and the electoral formula used to allocate seats is d'Hondt ($c = 1$)

Let us assume now that a party, p, expects to win 5 seats, $S_T^p = 5$. Given the uneven distribution of seats among the three districts and given that the number of seats that p can win in each district varies from 0 to M_d, there are several combinations of seats, $\mathbf{S}_j$, that produce the expected $S_T^p = 5$. These combinations are as follows:

$$\mathbf{S}_1 = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S}_2 = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix}$$

$$\mathbf{S}_3 = \begin{bmatrix} 4 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S}_4 = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}$$

$$\mathbf{S}_5 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$$

$$\mathbf{S}_6 = \begin{bmatrix} 3 & 0 & 2 \end{bmatrix}$$

$$\mathbf{S}_7 = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$$

$$\mathbf{S}_8 = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$$

$$\mathbf{S}_9 = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$$

$$\mathbf{S}_{10} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

$$\mathbf{S}_{11} = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

$\mathbf{S}_{12} = \begin{bmatrix} 0 & 3 & 2 \end{bmatrix}$

To calculate the aggregated threshold functions for each combination of seats we must first apply the individual threshold functions for each seat in each district. The following two tables provide this information.

Table 2.5: Necessary votes for each seat in each district

$\mathbf{M}_d$	$V^{nec}_{S_d=1}$	$V^{nec}_{S_d=2}$	$V^{nec}_{S_d=3}$	$V^{nec}_{S_d=4}$	$V^{nec}_{S_d=5}$
5	0.11	0.22	0.33	0.44	0.55
3	0.142	0.285	0.428		
2	0.166	0.33			

Table 2.6: Sufficient votes for each seat in each district

$\mathbf{M}_d$	$V^{suf}_{S_d=1}$	$V^{suf}_{S_d=2}$	$V^{suf}_{S_d=3}$	$V^{suf}_{S_d=4}$	$V^{suf}_{S_d=5}$
5	0.166	0.33	0.50	0.66	0.833
3	0.25	0.50	0.75		
2	0.33	0.66			

All this information is required to apply the aggregation method suggested above. The following two examples will show how this method works in more detail.

Example 3 $V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ *and* $V^{suf}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ *for* $S_1 = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$

A.- $V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^{D} \frac{M_d}{M}\left(V^{nec}_{S_d}\right) = \frac{5}{10} * 0.55 + \frac{3}{10} * 0 + \frac{2}{10} * 0 = \mathbf{0.275} = \mathbf{275}\ voters$$

B.- $V^{suf}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$

Table 2.7: Necessary votes

Voters	$\mathbf{M}_d$	S_{dj}	$V_{S_d}^{nec}$
500	5	5	0.55
300	3	0	0
200	2	0	0

Table 2.8: Sufficient votes

Voters	$\mathbf{M}_d$	S_{dj}	$V_{S_d}^{nec}$
500	5	5	0.833
300	3	0	0
200	2	0	0

$V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^{D} \frac{M_d}{M}\left(V_{S_d}^{suf}\right) = \frac{5}{10} * 0.833 + \frac{3}{10} * 0 + \frac{2}{10} * 0 = \mathbf{0.416} = \mathbf{416}$ *voters*

In other words, if party p expects to win 5 seats distributed according to $S_{1,}$ i.e. winning all the seats in the largest district, it knows that it must obtain a minimum of 27.5% of the vote. If the party p wins over 41.6% of the vote, it will obtain at least those 5 seats distributed as S_1 .

Example 4 $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ *and* $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ *for* $S_5 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$

A.- $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$

$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^{D} \frac{M_d}{M}\left(V_{S_d}^{nec}\right) = \frac{5}{10} * 0.33 + \frac{3}{10} * 0.142 + \frac{2}{10} * 0.166 = 0.165 + 0.0426 + 0.033 = \mathbf{0.2408} = \mathbf{240}$ *voters*

B.- $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$

Table 2.9: Necessary votes

Voters	$\mathbf{M}_d$	S_{djj}	$V_{S_d}^{nec}$
500	5	3	0.33
300	3	1	0.142
200	2	1	0.166

Table 2.10: Sufficient votes

Voters	$\mathbf{M}_d$	S_{dj}	$V_{S_d}^{nec}$
500	5	3	0.50
300	3	1	0.25
200	2	1	0.33

$$V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^{D} \frac{M_d}{M}\left(V_{S_d}^{suf}\right) = \frac{5}{10} * 0.50 + \frac{3}{10} * 0.25 + \frac{2}{10} * 0.33 = 0.25 + 0.075 + 0.066 = \mathbf{0.391} = \mathbf{391}\ voters.$$

The same method is applied to the remaining S_j combinations that produce $S_T = 5$. The following table shows the results for all the combinations.

As Table 2.11 shows, different $\mathbf{S}_j$ combinations require different shares of the vote in order to win the same S_T number of seats. A minimum criterion will be used in order to characterize different electoral systems. So, $\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ will refer to the minimum share of votes necessary to obtain S_T seats distributed according to a $\mathbf{S}_j$ combination. Equally, $\min V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ will refer to the minimum share of votes sufficient to get S_T distributed as $\mathbf{S}_j$.

One further observation should be made. The results obtained for the aggregated threshold functions do not tell us the exact number of seats that each party will win. As I noted in the previous section, threshold functions offer the best and the worst conditions in which a party will obtain a determined number of seats, but they

Table 2.11: Necessary and sufficient votes for all combinations that produce 5 seats

S_j	$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$	$V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$
$\begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$	0.275	0.416
$\begin{bmatrix} 4 & 1 & 0 \end{bmatrix}$	0.265	0.408
$\begin{bmatrix} 4 & 0 & 1 \end{bmatrix}$	0.255	0.40
$\begin{bmatrix} 3 & 2 & 0 \end{bmatrix}$	0.252	0.40
$\begin{bmatrix} 3 & 0 & 2 \end{bmatrix}$	0.233	0.38
$\begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$	0.242	0.391
$\begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$	0.239	0.391
$\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$	0.2287	0.381
$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$	0.220	0.375
$\begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$	0.217	0.375
$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$	0.207	0.366
$\begin{bmatrix} 0 & 3 & 2 \end{bmatrix}$	0.194	0.357

do not allocate seats to each party. As made clear here, the distribution of seats depends on how all the votes cast are distributed among all the parties, and this, of course, can only be known once the election has taken place. Threshold functions tell us about the conditions that allow a determined number of seats to be one in a specific electoral system.

This measure is nonetheless of considerable importance. Aggregated threshold functions tell us the range of votes that are required to win 50% of the seats in the parliament or just 1 seat. So, electoral systems can be characterized in accordance with these values. At a more practical level, aggregated threshold functions tell politicians about their chances of winning a determined number of seats in each district depending on the support they believe they have in the districts in question. This information may be extremely important when designing an electoral system, for example. These functions would also be of interest to political actors involved in the process of choosing an electoral systems since they could be used as key references for actors when making rational calculations. A knowledge of the percentage of the vote needed for, say, each majority-winning combination of seats in each electoral system would help political actors to decide which electoral system would give them the best chance of maximizing their presence in parliament given their real or estimated level of political support.

2.4 Data validation.

The main purpose of this section is to test the validity of the aggregated threshold functions. The idea here is to check whether the total share of the vote that produces the total number of seats won by political parties in a number of countries accords with the conclusions reached in the previous section. In order to carry out this test I have analyzed the electoral results obtained by the main political parties in one or more general elections in a number of countries. In order to consider as many possibilities as possible, I have chosen elections held under different electoral systems. In the case of divisor-based electoral systems, I have analyzed the results won by the main political parties in the 1979 general election in Spain, the 1991 and 1994 general elections in Bulgaria and the 1994 general election in Moldova. In order to test the validity of the aggregated threshold functions in quota-based electoral systems, I use the results won by the main political parties that participated in the 1986 general election in Costa Rica, the 1997 general election in Honduras and the 1999 general election in Benin. Finally, to test the validity of the functions in the case of majoritarian electoral systems, I will analyze the results of the 1993 general election in Canada and the 1997 general election in the United Kingdom. In total, these amount to 53 observations with sufficient variation to constitute a fair measure of the predictive capacity of aggregated threshold functions.

As I will explain in more detail below, the method used to validate the aggregated threshold functions is that described in the previous section. For each party there is a vector containing all votes won in each district and another vector with all the seats obtained in each district. If we add up all the shares of the vote won in each district, we obtain the total share of the vote for that party. The same operation is performed for the vectors for the seats. On the basis of this information and the remaining variables for each electoral system, I have applied $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ and $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$. As I will also discuss below, in order to test the functions and for practical reasons, I will use the effective number of parties[9] instead of the total number of competing parties[10].

The data validation for divisor-based electoral systems is shown in Table 2.12. As noted above, the countries to which aggregated threshold functions have been applied are Spain, Bulgaria and Moldova. These three countries are different from each other. Whereas Spain and Bulgaria have a large number of districts, 52 and 31 respectively, Moldova has a single district. The results given in Table 2.12 show how the share of votes won by each party, V_T^p, is contained in the interval formed by the necessary and the sufficient number of votes to win the number of seats obtained by each party, S_T^p. Formally,

$$V_T^p \in \left[V_{S_T}^{nec}, V_{S_T}^{suf}\right] \;\; \forall \; S_T^p \tag{2.29}$$

This idea must, however, be properly understood. What I intend to confirm here are conclusions 4, 5 and 6 referred above. Recall that conclusion 4 established that

[9]The effective number of parties, P, can be calculated using the following formula:

$P = \sum \frac{1}{(V_T^p)^2}$

where V_T^p is the total share of the vote for party p. This issue is discussed further in Chapter 4.

[10]It will be assumed that the effective number of parties is the same in all districts.

Table 2.12: Results for General Elections in three countries with divisor-based electoral systems.

Country	**Election**	**Party**	$V_T^p(\%)$	S_T^p	$V_{S_T}^{nec}$	$V_{S_T}^{suf}$
Spain	1979	UCD	34.84	168	31.96	41.27
Spain	1979	PSOE	30.4	121	24.71	30.62
Spain	1979	PCE	10.77	23	5.33	6.11
Spain	1979	CD	6.05	10	2.04	2.32
Spain	1979	CiU	2.69	8	1.89	2.14
Spain	1979	PNV	1.65	7	1.41	1.77
Spain	1979	PSA-PSA	1.81	5	1.06	1.29
Spain	1979	HB	0.96	3	0.63	0.77
Spain	1979	UN	2.11	1	0.26	0.28
Spain	1979	ERFN	0.69	1	0.26	0.28
Spain	1979	EE	0.48	1	0.20	0.25
Spain	1979	C-UPC	0.33	1	0.19	0.25
Spain	1979	PAR	0.21	1	0.20	0.25
Bulgaria	1991	SDS	34.36	110	33.37	40.92
Bulgaria	1991	BSP	33.14	106	31.36	39.04
Bulgaria	1991	DPS	7.55	24	6.94	8.76
Bulgaria	1994	BSPASEK	43.50	125	37.88	45.90
Bulgaria	1994	SDS	24.23	69	21.75	25.77
Bulgaria	1994	BZNS-DP	6.51	18	5.74	6.77
Bulgaria	1994	DPS	5.44	15	4.30	5.37
Bulgaria	1994	BBB	4.73	13	4.21	4.91
Moldova	1994	PDAM	43.18	56	41.18	53.33
Moldova	1994	PSMUE	22	28	20.59	26.67
Moldova	1994	BTI	9.21	11	8.09	10.48
Moldova	1994	AFPCD	7.53	9	6.62	8.57

Cases estimated correctly = 100%

$$\text{If } V_T^p < V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P), \text{then } S_T^p < S_T. \tag{2.30}$$

Consider the following illustration. Party PNV in the 1979 general election in Spain won 7 seats. The necessary proportion of votes required to win those 7 seats distributed exactly in the way that PNV won them is 1.41% of the vote at nationa level. Since PNV won 1.65% that minimun requirement was fulfilled.

Conclusion 5 showed something rather different. Formally,

$$\text{If } V_T^p > V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P), \text{then } S_T^p \geq S_T \tag{2.31}$$

What this expression indicates is the possibility for a party to win a higher number of seats when the total share of votes that it wins is higher than the the sufficient proportion of votes to win S_T seats distributed according to a particular combination $\mathbf{S}_j$. In other words, a higher share of vote won by party p may produce a different combination of seats that may also produce a higher number of the original total seats for which $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ was first applied. Look at information about CiU in Table 2.12. This party won 2.69% of the national vote and obtained 8 seats. The proportion of sufficient votes to win those 8 seats distributed in all districts exactly in the way that CiU won them is 2.14% that is to say, CiU won a share of vote 0.55% higher than the sufficient proportion of votes to win those 8 seats. However, CiU could only won 8 seats at most because given the distribution of those seats in the districts where they were won, the sufficient proportion of votes was 2.14%. CiU wasted then those 0.55% of the votes. This party could have won a different number of total seats if a different combination of seats were produced.

Finally, conclusion 6 established that,

$$\begin{aligned} \text{If } V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) &\leq V_T^p \leq V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P), \\ \text{then } \max(S_T^p) &= S_T \end{aligned} \tag{2.32}$$

This reasoning follows from the discussion just mentioned about the sufficient proportion of votes. Consider, once more, the electoral results obtained by the bulgarian political party BZNS-DP in the 1994 election. This party won 6.51% of the national vote and obtained 18 seats. When aggregated threshold functions are applied to the combination of seats that produced those 18 seats for this party it is observed that 5.74% is the necessary proportion of votes and 6.77% the sufficient proportion. The share of votes won by BZNS-DP is included in this interval and therefore the maximum number of seats that can be won are precisely those 18 seats distributed according to $\mathbf{S}_j$.

This logic can also be observed when other electoral systems are analyzed. Table 2.13 shows data for three different quota-based electoral systems. The countries included in this table all use the Hare quota electoral formula, but differ in terms of the number of districts and size of the assembly. So, whereas Costa Rica had 7 districts and an assembly with 57 members in the 1986 general election, Benin had 24 districts and an assembly of 84 members for the 1999 election and Honduras 18 districts which elect a total of 128 seats in the assembly. As in the case of divisor-based systems, the aggregated threshold values give us an interval in which all shares of the vote can be located. So, the R.B. party in the 1999 general election in Benin won 27 seats out of 84 seats with a 22.69% share of the total vote. Threshold functions predicted that that number of seats could not be won by a party obtaining less than 21.45% of the vote.

Finally, Table 2.14 shows the electoral results for two majoritarian electoral systems. Information is given for the 1997 general election in the United Kingdom (UK) and for the 1993 general election in Canada. The predictions of the aggregated threshold functions seem to be accurate at least in terms of the necessary number of votes to win S_T seats. No parties won S_T^P seats with a share of the vote below that predicted by the aggregated threshold functions.

Aggregated threshold functions seem, therefore, to constitute

Table 2.13: Results for General Elections in three countries with quota-based electoral systems

Country	Election	Party	$V_T^p(\%)$	S_T^p	$V_{S_T}^{nec}$	$V_{S_T}^{suf}$
Costa Rica	1986	LN	47.83	29	43.55	45.93
Costa Rica	1986	USC	41.44	24	34.78	37.15
Costa Rica	1986	PU	2.70	1	0.71	1.05
Costa Rica	1986	AL	0.32	1	0.71	1.05
Costa Rica	1986	UA	1.15	1	0.71	1.05
Costa Rica	1986	AP	2.43	1	0.71	1.05
Honduras	1997	PL	49.54	65	43.43	45.64
Honduras	1997	PN	41.56	56	35.47	37.82
Honduras	1997	PINU-SD	4.13	4	1.29	1.84
Honduras	1997	DC	2.61	2	0.64	0.92
Honduras	1997	UD	2.14	1	0.32	0.46
Benin	1999	RB	22,69	27	21.45	28.27
Benin	1999	PRD	12,17	11	6.61	11.57
Benin	1999	FARD	5,49	10	4.29	9.76
Benin	1999	PSD	9,27	9	6.41	8.84
Benin	1999	MADEP	9,21	6	2.80	5.62

Cases estimated correctly = 100%

Table 2.14: Results for General Elections in two majoritarian electoral systems.

Country	Election	Party	$V_T^P(\%)$	S_T^P	$V_{S_T}^{nec}$	$V_{S_T}^{suf}$
U.K.	1997	Labour	43.2	418	19.58	31.71
U.K.	1997	Conservatives	30.6	165	7.72	12.51
U.K.	1997	Lib-Dems	16.7	46	2.15	3.49
U.K.	1997	SNP	2	6	0.28	0.45
U.K.	1997	UU	0.83	10	0.46	0.75
U.K.	1997	SDLP	0.61	3	0.14	0.22
U.K.	1997	PC	0.51	4	0.18	0.30
Canada	1993	LP	41.3	177	15.33	30
Canada	1993	BQ	13.5	54	4.67	9.15
Canada	1993	RP	18.7	52	4.50	8.81
Canada	1993	NDP	6.9	9	0.77	0.33
Canada	1993	PCP	16.0	2	0.17326	0.33

Cases estimated correctly = 100%

a convincing measure through which to characterize electoral systems. In this chapter, I have described the method that comprise these functions. In the next chapter I will show the way in which they should be optimized.

Chapter 3

Optimization of Threshold Functions

This chapter continues the exploration of aggregated threshold functions. In this sense, it is necessary to consider the implications or the meanings of both the necessary and sufficient numbers of votes to win a given number of seats nationwide, $V_{S_T}^{nec}$, and $V_{S_T}^{suf}$ respectively. Reasons will be provided to justify the choice of $V_{S_T}^{nec}$ as the most appropriate measure to be used when characterizing electoral systems. Once this decision has been justified, the optimization of this function is the next step. Optimization implies finding the specific combination of seats that produces the minimum value of $V_{S_T}^{nec}$. By using the minimum value of the $V_{S_T}^{nec}$ function, we can establish the lowest point at which a complete electoral system allows a party to win one, or half of the seats in parliament, that is, the values used in this characterization. Optimization of $V_{S_T}^{nec}$ will be calculated for both divisor-based and quota-based methods, as well as for electoral systems with one tier of seat allocation and electoral systems with two tiers of seat allocation and independent electoral formulae.

3.1 Necessary or sufficient number of votes?

In the previous chapter threshold functions were discussed both at individual level and at aggregate level. Threshold functions were defined for elemental electoral systems. It will be remembered from the previous chapter that an elemental electoral system comprises three components: an electoral formula, c or n, a district magnitude, M_d, and the number of competing parties in that district, P_d. At the aggregate level other variables such as assembly size, the number of districts or the distribution of district magnitudes must be incorporated in order to estimate the aggregated threshold functions. To put it simply, aggregated threshold functions determine the necessary and the sufficient conditions to win a determined number of seats distributed in accordance with a particular combination of seats for any complete electoral system. On the basis of the discussion presented in the previous chapter it is possible to give a full definition of a complete electoral system:

Definition 5 *A complete electoral system is made of the following elements: an electoral formula, c or n, the number of districts into which the territory is divided, D, a $1xD$ vector with all district magnitudes, $\mathbf{M}_d$, the number of seats in the legislative assembly, M, and finally the number of parties*[1] *competing in all the districts, P.*

It should also be remembered that the purpose of defining these functions at the aggregate level is to elaborate a measure capable of predicting the necessary and the sufficient conditions to win a determined number of seats in any complete electoral system. These functions can, therefore, be used as a measure to characterize any complete electoral system on the basis of the following criteria:

a) The proportion of total votes required in a complete electoral systems to win a majority in parliament. Expressed in terms of a political party, p, a value of V_T^p such that $S_T^p = \dfrac{M}{2}$.

[1] It is assumed that the number of parties is the same in all districts.

b) The proportion of total votes required in a complete electoral systems to win a single seat in parliament. Or expressed in terms of a political party, p, a value of V_T^p such that $S_T^p = 1$.

However, further conceptual refinement is required in order to characterize electoral systems in accordance with the above criteria. More specifically, it must be decided whether to opt for either the aggregated function of necessary votes or the aggregated function of sufficient votes.

The choice between these two functions is not just a question of taste. For any measure to be parsimonious, it must combine simplicity and explanatory power. So, the simpler and the greater the substantive meaning of any measure, the more parsimonious it will be. The existence of two measures that offer different, though complementary, results for the same purpose, i.e. the shares of the vote necessary and sufficient to obtain a determined number of seats in all districts, would not appear to favour this goal. Hence, it is desirable to choose a single measure which can be used to characterize any complete electoral system. One possible measure for doing this could be some sort of average between both functions. Tables 2.12, 2.13 and 2.14 in the previous chapter show the necessary and sufficient proportion of votes to win a total number of seats distributed according to a particular combination of seat $\mathbf{S}_j^p$. As I disscussed previously, the share of votes to win these seats must lie in the interval

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^p, F, P) \leq V_T^p \leq V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j^p, F, P) \qquad (3.1)$$

Also recall that there are cases in which $V_T^p > V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j^p, F, P)$ and yet party p, obtain S_T seats distributed according to $\mathbf{S}_j^p$. As I explained, when this is the case, the difference between these two shares of votes should be considered as wasted votes by party p, since to win those S_T seats distributed according to $\mathbf{S}_j^p$, it would have sufficed to win just $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j^p, F, P)$.

All this may lead us to conclude, then, that one possible measure could be obtained by finding an intermediate point between the two extrema values. However this option should be rejected for the following reasons. The total number of seats won by a party, S_T^p, depends on the votes won, V_T^p, but also on the votes won by the other competing parties. For this reason there will be cases in which V_T^p will be closer to $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^p, F, P)$ or to $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j^p, F, P)$ or will simply be in the mid-point between these two value. If we use a unified measure based on some sort of average, the expected value would not match the observed value because the latter can only be known after the elections have taken place. What makes aggregated threshold functions valuable is their capability to predict *ex ante* the bounds within which a determined number of seats will be won in a specific institutional framework. As already noted, aggregated threshold values do not tell us the exact share of votes required to win S_T seats before the elections take place because that information can only be obtained once the distribution of all votes among all parties is known, i.e. once the elections has been held. It is not necessary, therefore, to use an average measure based on both aggregated threshold functions.

There is another reason why such a measure would be inappropriate. Both $V_{S_T}^{nec}$ and $V_{S_T}^{suf}$ have specific meaning of their own which that makes them interesting and appealing. $V_{S_T}^{nec}$ refers, as explained above, to the condition that must be fulfilled in order to obtain S_T seats, while $V_{S_T}^{suf}$ refers to the sufficient condition to win those S_T seats. They are functions that also have a robust substantive meaning. For example, a party that wins a share of votes below $V_{S_T}^{nec}$ knows that it will not have a chance of winning those S_T, presuming all other variables remain constant. Remember the previous discussion of Lijphart's effective threshold (Lijphart 1994). This is a measure that is calculated for the sole purpose of fitting expected values to the observed values. As I argued above, this is an empirical measure that lacks any substantive meaning. As a result, it is a measure which does not provide the grounds for conclusive interpretation. Applying the same logic, an average measure based on threshold functions would be meaningless because no substantive explanation could be attached to it. And this is another reason to reject this kind of measure.

Hence, the only option available is to select either of the aggregated threshold functions. Since both functions have a clear substantive meaning, the option for one or the other will determine the flexibility of the classification. If $V_{S_T}^{suf}$ is used as a measure to characterize a complete electoral system, then a maximal criterion will be used; on the other hand, if $V_{S_T}^{nec}$ is used for that purpose, a minimal criterion will be adopted. This can be seen mathematically by observing that $V_{S_T}^{suf} > V_{S_T}^{nec}$ or

$$\sum_{d=1}^{D} \frac{M_d}{M} \left(\frac{S_d - 1 + c}{M_d + 1 + P(c-1)} \right) \geq \sum_{d=1}^{D} \frac{M_d}{M} \left(\frac{S_d - 1 + c}{M_d - 1 + Pc} \right) \quad (3.2)$$

Solving for P we find that

$$P \geq 2 \quad (3.3)$$

which is always true since by definition $P \geq 2$

Using $V_{S_T}^{suf}$ to classify a complete electoral system would mean using the most extreme value, or the sufficient condition, required to obtained a particular number of seats, whereas $V_{S_T}^{nec}$ offers the least extreme value, or the necessary condition, required to obtain that number of seats.

As noted above, In this research, I will use the necessary number of votes nationwide, $V_{S_T}^{nec}$ to characterize any complete electoral systems. This decision has been taken for the following reasons. Firstly, using $V_{S_T}^{nec}$ instead of $V_{S_T}^{suf}$ means adopting an exclusive criterion: any party that obtains a share of votes below $V_{S_T}^{nec}$ will have no chance of obtaining S_T seats *ceteris paribus*. This criterion is attractive given that one of the objectives of characterizing complete electoral systems is to identify the points at which the institutional framework makes it possible to win a determined number of seats such as $\frac{M}{2}$. $V_{S_T}^{nec}$, therefore, does not guarantee the share of the vote required to win S_T seats, but rather establishes the limit under which S_T will never be won. Secondly, $V_{S_T}^{nec}$ decreases the uncertainty about S_T that is implicit in $V_{S_T}^{suf}$. If a party, p, pursues S_T seats distributed according to the vector $\mathbf{S}_j$ and wins a share of votes higher than $V_{S_T}^{suf}$, then p knows for sure that S_T seats distributed according to $\mathbf{S}_j$ will be won; but the party also knows that this is the minimum number of seats that can be won given their votes. In other words, there is some uncertainty about the real number of total seats that p will win, as well as about the distribution of these. When using $V_{S_T}^{nec}$, this uncertainty is reduced in the following sense: if a party, p, expects to win S_T seats distributed according to $\mathbf{S}_j$, then p knows that to have a chance of winning exactly this number of S_T seats distributed according to $\mathbf{S}_j$, its share of the vote has to be greater than $V_{S_T}^{nec}$. Party p knows that its ability to win these S_T seats also depends on the other parties' performance in the elections, and hence that there is also some degree of uncertainty about its final result. However this political party, p, can be certain that if $V_T^p > V_{S_T}^{nec}$ then its expectations about S_T seats distributed according to S_j will be fulfilled.

3.2 Optimizing $V_{S_T}^{nec}$ for electoral systems with 1 tier of seat allocation.

In the previous section I pointed out that the two criteria used to characterize any complete electoral system are the values of $V_{S_T}^{nec}$ that would produce $S_T = \frac{M}{2}$ and $S_T = 1$. But what exactly are these values? As shown in the previous chapter, the same total number of seats, $S_{T,}$ can be obtained through different combinations of seats, $\mathbf{S}_j$, $\mathbf{S}_k$...$\mathbf{S}_z$ so that,

$$\sum_{d=1}^{D} S_{d_j} = \sum_{d=1}^{D} S_{d_k} = ... = \sum_{d=1}^{D} S_{d_z} = S_T \tag{3.4}$$

Since these particular combinations of seats produce different results when the $V_{S_T}^{nec}$ function is applied, which of them should be used to characterize a complete electoral system? For the purpose of this research, I will use the combination that produces the minimum values that fulfills the above criteria. The reason for doing so is simple: the adoption of the minimum value that produces, for example, a majority in the parliament is a *sine qua non* condition to win that number of seats. That is to say, if a criterion for classifying any complete electoral system is to identify the point at which that complete electoral system makes it possible for a party to win a parliamentary majority, then the minimum value of $V_{S_T}^{nec}$ establishes the minimum point that allows that criterion to be met. Hence, by discovering the particular combination of seats that produces the minimum number of necessary votes to win a majority in the parliament a minimum condition is established. Recapitulating, the value of $\min V_{S_T}^{nec}$ will be calculated both for the minimum share of the vote which a party must win to obtain a majority in parliament, and the minimum share of the vote which a party must win to obtain a single seat in each complete electoral system.

In the rest of this section I will focus on electoral systems with just 1 tier of seat allocation. In these electoral systems, seats are only allocated at district level, regardless of the number and size of the districts. In the next section, some ideas about optimizing $V_{S_T}^{nec}$ in mixed electoral systems with 2 tiers of seat allocation and independent electoral formulae will be considered.

3.2.1 Optimizing $V_{S_T}^{nec}$ for $S_T = 1$.

As noted above, one of the criterion that can be used to characterize a complete electoral system is the share of the vote required to obtain the minimum level of parliamentary representation. The literature on electoral systems includes an interesting discussion about the location of this minimum threshold (Lijhart and Gibberd 1977; Taagepera and Shugart 1989; Lijphart 1994; Penadés 2000). The overwhelming majority of scholars favours using the effective threshold as presented by Lijphart and Taagepera. As I pointed out in the previous chapter, this measure suffers from a number of limitations. One of these is that it only refers to the district level. Aggregated threshold functions enable us to identify the cost, in terms of share of the vote, of winning a single seat in different districts.

Optimizing $V_{S_T}^{nec}$ for $S_T = 1$ for divisor-based electoral formulae.

The method used to identify the district where 1 seat is won with the minimum amount of votes is shown in the following theorem.

Theorem 1 *Given a complete electoral system with a divisor-based electoral formula*[2]*, c, where the seats in the parliament, M, are distributed unevenly among all districts, the combination of seats that produce* $\min V_{S_T}^{nec}$, $\mathbf{S}_j^*$, *for $S_T = 1$ is the one where the seat is won in the smallest district.*

Proof. Let CES stands for a complete electoral systems with a parliament of size M, a number of districts, D, a distribution of districts

$\mathbf{M}_d = \begin{bmatrix} M_1, & M_2, & ..., & M_D \end{bmatrix}$ restricted to $\sum_{d=1}^{D} M_d = M$, a number of parties P and a divisor-based electoral formula, c. Assume that $M_d \leq ... \leq M_D$ and be any two consecutive district magnitudes such that

$$M_{d+1} = M_d + \Delta_d \tag{3.5}$$

[2] Assume that c can take only two values, $c = 0.5$ and $c = 1$ which correspond to systems using Säinte-Lague and d'Hondt formulae respectively.

where $\Delta_d \geq 0$ refers to the magnitude difference between M_d and M_{d+1}.

Let $S_T = 1$ be assigned to a distribution of seats $\mathbf{S}_j^*$ where the single seat is won in district M_d. The value of $V_{S_T}^{nec}$ for this particular distribution of seats is

$$\frac{M_d}{M}\left[\frac{c}{M_d - 1 + Pc}\right] \tag{3.6}$$

Likewise, let $\mathbf{S}_k$ be a particular distribution of seats also producing $S_T = 1$ but where the single seat is won in district M_{d+1} instead of in M_d. Using 3.5 we express the value of $V_{S_T}^{nec}$ for $\mathbf{S}_k$ as

$$\frac{M_d + \Delta_d}{M}\left[\frac{c}{M_d + \Delta_d - 1 + Pc}\right] \tag{3.7}$$

For theorem 1 to hold it must be the case that

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) \geq V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P) \tag{3.8}$$

So from expressions 3.6 and 3.7 we get

$$\frac{M_d + \Delta_d}{M}\left[\frac{c}{M_d + \Delta_d - 1 + Pc}\right] \geq \frac{M_d}{M_d - 1 + Pc} \tag{3.9}$$

Solving for P the following inequality is obtained

$$P \geq \frac{1}{c} \tag{3.10}$$

When $c = 1$, $P \geq 1$, which is something that is true by definition and when $c = 0.5$, $P \geq 2$, which is also true since by definition $P \geq 2$. If this is true for any two rank-ordered consecutive districts then, by induction, it must be true that $\mathbf{S}_j^*$ does produce $\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$ since

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) \geq V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P) \tag{3.11}$$

■

Optimizing $V_{S_T}^{nec}$ for $S_T = 1$ for quota-based electoral formulae.

The following theorem shows the district in which $\min V_{S_T}^{nec}$ for $S_T = 1$ is obtained in a complete electoral system with a quota-based electoral formula.

Theorem 2 *Given a complete electoral system with a quota-based electoral formula*[3], n, *and where the number of seats in the parliament,* M, *is distributed unevenly among all districts, the combination of seats that produces* $\min V_{S_T}^{nec}$, $\mathbf{S}_j^*$, *for* $S_T = 1$ *is the following:*

a) when $n = 1$, *the seat is won in the smallest district.*

b) when $n = 0$, *the district where the seat is won is irrelevant since the value required to obtain* 1 *seat is the same in all districts.*

Proof . Let CES stand for a complete electoral system with a parliament of size M, a number of districts, D, a distribution of districts $\mathbf{M}_d = \left[\begin{array}{cccc} M_{1,} & M_2, & ..., & M_D \end{array}\right]$ restricted to $\sum_{d=1}^{D} M_d = M$, a number of parties, P and a quota-based electoral formula, n. Assume that $M_d \leq ... \leq M_D$ and be any two consecutive districts such that

$$M_{d+1} = M_d + \Delta_d \tag{3.12}$$

where $\Delta_d \geq 0$ refers to the magnitude difference between M_d and M_{d+1}.

Let $S_T = 1$ be distributed in $\mathbf{S}_j^*$ where the single seat is won in district M_d. The value of $V_{S_T}^{nec}$ for this particular distribution of seats is

$$\frac{M_d}{M}\left[\frac{n+1}{P\left(M_d+n\right)}\right] \tag{3.13}$$

Likewise, let $\mathbf{S}_k$ be a particular distribution of seats also producing $S_T = 1$, but where the single seat is won in district M_{d+1} instead

[3] Assume that n can only take two values $n = 0$ and $n = 1$ which correspond to the Hare quota and the Droop quota respectively.

of in M_d. The value of $V^{nec}_{S_T}$ for $\mathbf{S}_k$ is according to 3.12

$$\frac{M_d + \Delta_d}{M}\left[\frac{n+1}{P\,(M_d + \Delta_d + n)}\right] \tag{3.14}$$

For theorem 2 to be true it must be the case that

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_k, n, P) \geq V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, n, P) \tag{3.15}$$

So from expressions 3.13 and 3.14 we get

$$\frac{M_d + \Delta_d}{M}\left[\frac{n+1}{P\,(M_d + \Delta_d + n)}\right] \geq \frac{M_d(n+1)}{M_d + n} \tag{3.16}$$

This expression can be simplified as

$$\Delta_d(n^2 + n) \geq 0 \tag{3.17}$$

When $n = 1$, then,

$$2\Delta_d \geq 0 \tag{3.18}$$

which is always true since the difference between two consecutive district magnitudes must always be a positive integer. When $n = 0$,

$$\Delta_d(n^2 + n) = 0 \tag{3.19}$$

which is also always true.

Hence, if this is true for any two consecutive rank-ordered districts, then by induction

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) = V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, c, P) \tag{3.20}$$

■

Table 3.1 shows results that confirm theorems 1 and 2. So, for a distribution of districts,

$$\mathbf{M}_d = \begin{bmatrix} 70 & 15 & 5 & 5 & 5 \end{bmatrix} \tag{3.21}$$

Table 3.1: Combinations of districts and seats for c, n, M=100, D=5 and D=8

M = 100, P = 3, D = 5 and D = 8, S$_T$= 1				
$\mathbf{M}_d$				
70 15 5 5 5	$c = 1$	$c = 0.5$	$n = 0$	$n = 1$
$\mathbf{S}_j$	$V_{S_T}^{nec}$	$V_{S_T}^{nec}$	$V_{S_T}^{nec}$	$V_{S_T}^{nec}$
1 0 0 0 0	0.009	0.005	**0.003**	0.0068
0 1 0 0 0	0.008	0.0048	**0.003**	0.0062
0 0 0 0 1	**0.007**	**0.0045**	**0.003**	**0.005**
$\mathbf{M}_d$				
50 15 15 4 4 4 4 4				
$\mathbf{S}_j$				
1 0 0 0 0 0 0 0	0.009	0.005	**0.003**	0.0065
0 0 1 0 0 0 0 0	0.008	0.0048	**0.003**	0.0062
0 0 0 0 0 0 0 1	**0.0067**	**0.0044**	**0.003**	**0.005**

it can be seen that the minimum value for $V_{S_T}^{nec}$ is obtained when the seat is won in the smallest district,

$$\mathbf{S}_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.22}$$

which is what theorem 1 and part a) of theorem 2 show.

Note also how for any given distribution of districts the value of winning 1 seat is always the same for $n = 0$, regardless of the size of the district.

3.2.2 Optimizing $V_{S_T}^{nec}$ for $S_T = \dfrac{M}{2}$ using divisor-based electoral formulae.

I will now find the optimal distribution of seats that produces $\min V_{S_T}^{nec}$. Since $\dfrac{M}{2}$ is a much larger number than 1, some initial intuitions might be of some help. From Tables 3.2, 3.3 and 3.4 it can be seen that most of the combinations that produce $\min V_{S_T}^{nec}$ are those where the S_T seats are primarily won in the smallest districts, then in middle-size districts and finally in the biggest districts.

One way of seeing why the optimum distribution of seats, S_j^*, could be located in the smallest districts is by using differential calculus. From

$$V_{S_T}^{nec} = \sum_{d=1}^{D} \frac{M_d}{M}(V_{S_d}^{nec}) \tag{3.23}$$

the element of that sum for district d is

$$\frac{M_d(S_d - 1 + c)}{M(M_d - 1 + Pc)}, \tag{3.24}$$

assume that $S_d = 1$, and partial differentiation with respect to M_d is taken,

$$\frac{\partial}{\partial M_d}\frac{M_d c}{M(M_d - 1 + Pc)} \tag{3.25}$$

Solving this partial derivative the following is obtained:

$$\frac{c}{M}\frac{1}{(M_d - 1 + Pc)^2}(Pc - 1) \geq 0 \tag{3.26}$$

The first conclusion to be drawn from this partial derivative is that the $V_{S_T}^{nec}$ curve does not decrease for every point in its domain. That can be seen in Figure 3.1.

If the curve increases continuously in its domain, it means that the lowest values of the curve are found in the lowest district magnitudes. So, for $S_d = 1$, $V_{S_T}^{nec}$ increases as the size of M_d does. Figure 3.1 illustrates these ideas. The two lines indicate the values of $V_{S_T}^{nec}$ when $S_d = 1$ and $1 \leq M_d \leq 100$. The solid line shows the values when $c = 1$ and the dotted line when $c = 0.5$. Both curves show how the lowest value of $V_{S_T}^{nec}$ is obtained when the size of the district is small.

On the basis of all these intuitions, the following two theorems can be defined.

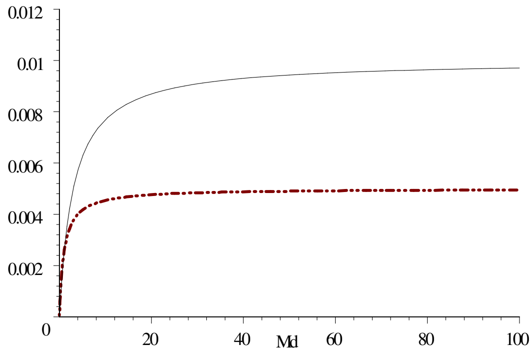


Figure 3.1: $V_{S_T}^{nec}$ for $S_d = 1$, $M = 100$, $P = 3$, c and M_d

Theorem 3 *Given a complete electoral system with a divisor-based electoral formula, c, where the number of seats in the parliament, M, is distributed unequally among all districts, the combination of seats that produces* $\min V_{S_T}^{nec}$, $\mathbf{S}_j^*$, *for* $S_T = \dfrac{M}{2}$ *depends on the value of c.*

a) If $c = 1$, *then the combination of seats that produces* $\min V_{S_T}^{nec}$, $\mathbf{S}_j^*$, *is one in which the* S_T *seats are all distributed in small districts. More specifically, seats will be distributed filling up first the smallest districts, then the second smallest district and so on until the total number of* S_T *is reached.*

b) For $c = 0.5$ *the combination of seats that produces* $\min V_{S_T}^{nec}$, $\mathbf{S}_j^*$ *is one in which* S_T *must be distributed among all districts in* $\mathbf{M}_{d.}$ *Seats are distributed as in a), with the smallest districts being filled until completed, and then filling up bigger districts. However, none of these bigger districts can be without representation. Bigger districts must have at least 1 seat each.*

Proof of a). Let CES stands for a complete electoral systems with a parliament of size M, a number of districts, D, a distribution of districts $\mathbf{M}_d = \begin{bmatrix} M_1, & M_2, & .., & M_D \end{bmatrix}$, a number of parties, P, and

Table 3.2: Combinations of districts and seats for c, M=100 and D=5

M = 100, P = 3, D = 5, S_T= 50		
$\mathbf{M}_d$		
20 20 20 20 20	$c = 1$	$c = 0.5$
$\mathbf{S}_j$	$V_{S_T}^{nec}$	$V_{S_T}^{nec}$
0 0 10 20 20	0.4545	0.4732
1 1 8 20 20	0.4545	0.4634
10 10 10 10 10	**0.4545**	**0.4634**
$\mathbf{M}_d$		
70 15 5 5 5		
$\mathbf{S}_j$		
50 0 0 0 0	0.4861	0.4915
35 15 0 0 0	0.4726	0.4829
20 15 5 5 5	**0.4339**	**0.4567**
M_D		
50 20 20 5 5		
$\mathbf{S}_j$		
50 0 0 0 0	0.4808	0.4901
1 19 20 5 5	0.4356	**0.4575**
0 20 20 5 5	**0.4351**	0.4623

a divisor-based electoral formula[4], c. Assume that $M_d \leq ... \leq M_D$ restricted to $\sum_{d=1}^{D} M_d = M$. District magnitudes are rank-ordered, so two consecutive districts can be expressed as

$$M_{d+1} = M_d + \Delta_d \tag{3.27}$$

where $\Delta_d \geq 0$ refers to the magnitude of the difference between M_d and M_{d+1}.

[4] Assume that c can only take two values $c = 0.5$ and $c = 1$ which corresponds to the Säinte-Lague and d´Hondt formular respectively.

Table 3.3: Combinations of districts and seats for c, M=100 and D=8

M = 100, P = 3, D = 8, $\mathbf{S}_T$**= 50**		
$\mathbf{M}_d$		
13 13 13 13 12 12 12 12	$c = 1$	$c = 0.5$
$\mathbf{S}_j$	$V^{nec}_{S_T}$	$V^{nec}_{S_T}$
13 13 13 11 0 0 0 0	0.4333	0.4622
1 1 1 1 10 12 12 12	0.4290	**0.4417**
0 0 0 2 12 12 12 12	**0.4288**	0.4560
$\mathbf{M}_d$		
50 15 15 4 4 4 4 4		
$\mathbf{S}_j$		
50 0 0 0 0 0 0 0	0.4808	0.4901
1 14 15 4 4 4 4 4	0.3988	**0.4315**
0 15 15 4 4 4 4 4	**0.3980**	0.4362
$\mathbf{M}_d$		
20 20 20 20 5 5 5 5		
$\mathbf{S}_j$		
20 20 10 0 0 0 0 0	0.4545	0.4732
1 1 8 20 5 5 5 5	0.4156	**0.4368**
0 0 10 20 5 5 5 5	**0.4156**	0.4456

Suppose there is a distribution of seats $\mathbf{S}^*_j$ that produces a total number of seats S_T. $\mathbf{S}^*_j$ is a $1xD$ vector that shows the number of seats that a party, $p \in \mathbf{P}$, wins in each district. In this particular distribution of seats, S_d seats are won in district M_d. Remember that

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, c, P) = \sum_{d=1}^{D} \frac{M_d}{M} V^{nec}_{S_d} \tag{3.28}$$

So the necessary number of votes to win S_d seats is

$$\frac{M_d}{M}\left[\frac{S_d - 1 + c}{M_d - 1 + Pc}\right] \tag{3.29}$$

Table 3.4: Combinations of districts and seats for c, M=100 and D=20

M = 100, P = 3, D = 20, $\mathbf{S}_T$**= 50**		
$\mathbf{M}_d$		
50 5 5 5 5 5 5 5 4 1 1 1 1 1 1 1 1 1 1 1	$c = 1$	$c = 0.5$
$\mathbf{S}_j$	$V^{nec}_{S_T}$	$V^{nec}_{S_T}$
50 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0.4808	0.4901
1 4 5 5 5 5 5 5 4 1 1 1 1 1 1 1 1 1 1 1	0.3158	**0.35**
0 5 5 5 5 5 5 5 4 1 1 1 1 1 1 1 1 1 1 1	**0.3133**	0.3541
$\mathbf{M}_d$		
20 20 13 12 11 3 3 2 2 2 2 2 2 1 1 1 1 1 1 1		
$\mathbf{S}_j$		
20 20 10 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0.4503	0.4720
1 1 1 12 11 3 3 2 2 2 2 2 2 1 1 1 1 1 1 1	0.3354	**0.3549**
0 0 3 12 11 3 3 2 2 2 2 2 2 1 1 1 1 1 1 1	**0.3346**	0.3644

Now suppose that there is another distribution of seats $\mathbf{S}_k$ that also produces the same total number of seats, S_T. In this new distribution $S_d - 1$ seats are won in district M_d and 1 seat is won in district M_{d+1}. Using expression 3.27 the necessary number of votes to win S_d seats is

$$\frac{M_d}{M}\left[\frac{(S_d-1)-1+c}{M_d-1+Pc}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{c}{M_d+\Delta_d-1+Pc}\right] \quad (3.30)$$

For Theorem 3 to be true the following must hold

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) \geq V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, c, P) \quad (3.31)$$

Using expressions 3.27, 3.29 and 3.30, expression 3.31 can be simplified as

$$\frac{M_d c+\Delta_d c}{M_d+\Delta_d-1+Pc} \geq \frac{M_d}{M_d-1+Pc} \quad (3.32)$$

If $c = 1$, then

$$\frac{M_d + \Delta_d}{M_d + \Delta_d - 1 + P} \geq \frac{M_d}{M_d - 1 + P} \tag{3.33}$$

Solving for P

$$P \geq 1 \tag{3.34}$$

which is always true by definition. So by induction it can be said that $\mathbf{S}_j^*$ produce $\min V_{S_T}^{nec}(M, D, \mathbf{S}_j^*, c, P)$ since

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) > V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P) \tag{3.35}$$

■

Proof of b). Let CES stands for a complete electoral systems with a parliament of size M, a number of districts, D, a distribution of districts $\mathbf{M}_d = \begin{bmatrix} M_1, & M_{2,} & ..., & M_D \end{bmatrix}$, a number of parties, P and a divisor-based electoral formula, c. Assume that $M_d \leq ... \leq M_D$ restricted to $\sum_{d=1}^{D} M_d = M$. Since district magnitudes are rank-ordered two consecutive districts can be expressed as

$$M_{d+1} = M_d + \Delta_d \tag{3.36}$$

where $\Delta \geq 0$ refers to the magnitude difference between M_d and M_{d+1}.

Suppose there is a distribution of seats $\mathbf{S}_k$ that produces a total number of seats S_T. $\mathbf{S}_k$ is a $1xD$ vector that shows the number of seats that a party, $p \in \mathbf{P}$, wins in each district. In this distribution S_d seats are won in district M_d. The necessary number of votes to win these S_d seats is

$$\frac{M_d}{M}\left[\frac{S_d - 1 + c}{M_d - 1 + Pc}\right] \tag{3.37}$$

Now suppose that there is another distribution of seat $\mathbf{S}_j^*$ that also produces the same total number of seats, S_T. In this new distribution $S_d - 1$ seats are won in district M_d and 1 seat is won in district M_{d+1}. Using expression 3.36 the necessary number of votes to win these S_d seats is

$$\frac{M_d}{M}\left[\frac{(S_d-1)-1+c}{M_d-1+Pc}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{c}{M_d+\Delta_d-1+Pc}\right] \quad (3.38)$$

For part b) of theorem 3 to be true the following must hold

$$V_{S_T}^{nec}(M,\mathbf{M}_d,\mathbf{S}_k,c,P) \geq V_{S_T}^{nec}(M,\mathbf{M}_d,\mathbf{S}_j^*,c,P) \quad (3.39)$$

Using expressions 3.36, 3.37 and 3.38, expression 3.39 can be simplified as

$$\frac{M_d}{M_d-1+Pc} \geq \frac{M_dc+\Delta_dc}{M_d+\Delta_d-1+Pc} \quad (3.40)$$

When $c=0.5$, then

$$\frac{2M_d}{2M_d-2+P} \geq \frac{M_d+\Delta_d}{2(M_d+\Delta_d)-2+P} \quad (3.41)$$

If $P=2$, expression 3.41 produces

$$\frac{M_d}{M_d} \geq \frac{M_d+\Delta_d}{2(M_d+\Delta_d)} \quad (3.42)$$

which is always true. This inequality holds for $P \geq 2$ as required.

If this is true for any two consecutive rank-ordered districts, then, by induction it can be said that $\mathbf{S}_j^*$ produce $\min V_{S_T}^{nec}(M,D,\mathbf{S}_j^*,c,P)$ since

$$V_{S_T}^{nec}(M,\mathbf{M}_d,\mathbf{S}_k,c,P) \geq V_{S_T}^{nec}(M,\mathbf{M}_d,\mathbf{S}_j^*,c,P) \quad (3.43)$$

■

Theorem 3 offers us the method to be used to find the combination of seats that produces the minimum value of $V_{S_T}^{nec}$ for $S_T=\frac{M}{2}$. Let us consider a number of examples that serve to illustrate these conclusions. As Table 3.2 shows, for a distribution of districts,

$$\mathbf{M}_d = \left[\begin{array}{ccccc} 50 & 20 & 20 & 5 & 5 \end{array}\right] \quad (3.44)$$

the combination of seats, $\mathbf{S}_j^*$, that produces $\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$ for $c = 1$ is

$$\mathbf{S}_j^* = \left[\begin{array}{ccccc} 0 & 20 & 20 & 5 & 5 \end{array}\right] \tag{3.45}$$

however, when $c = 0.5$ the combination of seats, $\mathbf{S}_j^*$, that produces
$\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$ is

$$\mathbf{S}_j^* = \left[\begin{array}{ccccc} 1 & 19 & 20 & 5 & 5 \end{array}\right] \tag{3.46}$$

Note how a variation of just 1 in seats in vectors 3.46 and 3.45 produces a different combination of seats that result in the lowest value of $V_{S_T}^{nec}$ in each $\mathbf{S}_j^*$ depending on the value of c. The result is logical and coherent, since in more proportional electoral systems the cost of winning the first seat is lower. One more example shows this relationship. Imagine the following distribution of districts from Table 3.4

$$\mathbf{M}_d = [20\ 20\ 13\ 12\,11\ 3\ 3\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1] \tag{3.47}$$

For $c = 1$, the combination of seats, $\mathbf{S}_j^*$, that produces $\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$ is

$$\mathbf{S}_j^* = [0\ 0\ 3\ 12\,11\ 3\ 3\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1] \tag{3.48}$$

which is exactly what theorem 3 a) says. For $c = 0.5$ the combination of seats, $\mathbf{S}_j^*$, that produces $\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$ is

$$\mathbf{S}_j^* = [1\ 1\ 1\ 12\,11\ 3\ 3\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1] \tag{3.49}$$

A result that,again, is explained by theorem 3 b).

Theorem 3 refers to the case in which the district magnitudes are unequal. Do the same conclusions hold as in the case in systems in which district magnitudes are equal? What is the combination of seats, $\mathbf{S}_j^*$, that produces $\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$? The following theorem casts some light on these questions.

Theorem 4 *Given a complete electoral system where the number of seats in the parliament, M, is distributed evenly, i.e. $M_d = M_D$ for all $M_d \in \mathbf{M}_d$, then for $S_T = \dfrac{M}{2}$ the following is true:*

a) The value of $V_{S_T}^{nec}$ is the same for all possible combinations of seats that produce S_T when $c = 1$.

b) However, if $c = 0.5$, then, the value of $V_{S_T}^{nec}$ is the same for all possible combinations of seats that produce S_T if all seats are distributed among all districts, $S_d^* > 0$ for all $S_d^* \in \mathbf{S}_j^*$. In other words, all particular combinations of seats that do not include a seat in any of the districts produce a higher value of $V_{S_T}^{nec}$.

Proof of a). Again, let CES stand for a complete electoral systems with a parliament of size M, a number of districts, D, a distribution of districts $\mathbf{M}_d = \begin{bmatrix} M_1, & M_2, & ..., & M_D \end{bmatrix}$, a number of parties, P, and a divisor-based electoral formula, c. Assume that $M_d \leq ... \leq M_D$ restricted to $\sum_{d=1}^{D} M_d = M$. District magnitudes are rank-ordered, so two consecutive districts can be expressed as

$$M_{d+1} = M_d + \Delta_d \qquad (3.50)$$

where $\Delta_d \geq 0$ refers to the magnitude difference between M_d and M_{d+1}.

Suppose there is a distribution of seats $\mathbf{S}_j^*$ that produces a total number of seats S_T. $\mathbf{S}_j^*$ is a $1xD$ vector that shows the number of seats that a party, $p \in \mathbf{P}$, wins in each district. Assume that S_T are distributed evenly between all the equal size districts. In other words, each element of the vector $\mathbf{S}_j^*$ is equal, $S_d^* = ... = S_D^*$. The value of $V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, c, P)$ for this distribution is the following:

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, c, P) = \sum_{d=1}^{w} \frac{M_d}{M}(V^{nec}_{S^*_{dj}}) = \frac{M_d}{M}\left\lceil \frac{\sum_{d=1}^{D} S^*_d - D + Dc}{M_d - 1 + Pc} \right\rceil \tag{3.51}$$

And the necessary number of votes to win S_d seats is

$$\frac{M_d}{M}\left\lceil \frac{S_d - 1 + c}{M_d - 1 + Pc} \right\rceil \tag{3.52}$$

Now suppose another distribution of seats, $\mathbf{S}_k$, that also produces the same total number of seats, $S_{T.}$ In this new distribution $S_d - 1$ seats are won in district M_d and 1 seat is won in district M_{d+1}. The necessary number of votes to win S_d seats in this distribution is

$$\frac{M_d}{M}\left\lceil \frac{(S_d - 1) - 1 + c}{M_d - 1 + Pc} \right\rceil + \frac{M_d + \Delta_d}{M}\left\lceil \frac{c}{M_d + \Delta_d - 1 + Pc} \right\rceil \tag{3.53}$$

Since all districts have the same magnitude, $\Delta_d = 0$. So the expression 3.53 can be simplified as

$$\frac{M_d}{M}\left\lceil \frac{S_d - 2 + 2c}{M_d - 1 + Pc} \right\rceil \tag{3.54}$$

For part a) of theorem 4 to hold true it must be the case that

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, c, P) = V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) \tag{3.55}$$

Using expressions 3.50, 3.52 and 3.54, expression 3.55 can be simplified to

$$S_d - 2 + 2c = S_d - 1 + c \tag{3.56}$$

and solving for c

$$c = 1 \tag{3.57}$$

as required. If this is true for any two consecutive rank-ordered districts, then, by induction it must also be true that that $\mathbf{S}_j^*$ produces $\min V_{S_T}^{nec}(M, D, \mathbf{S}_j^*, c, P)$ since

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) \geq V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P) \tag{3.58}$$

■

Proof of b). This proof has two parts. The first part serves to prove that any combination of seats where $S_d^* = 0$ for any $S_d^* \in \mathbf{S}_j^*$ produces a higher value of $V_{S_T}^{nec}$. In the second part of the proof it is shown that as in case a) of theorem 4 when $c = 0.5$ all combinations of seats where $S_d^* > 0$ for all $S_d^* \in \mathbf{S}_j^*$ produce the same value for $V_{S_T}^{nec}$.

Let CES stands for a complete electoral systems with a parliament of size M, a number of districts, D, a distribution of districts $\mathbf{M}_d = \begin{bmatrix} M_1, & M_2, & ..., & M_D \end{bmatrix}$, a number of parties P and a divisor-based electoral formula, c. Assume that $M_d \leq ... \leq M_D$ restricted to $\sum_{d=1}^{D} M_d = M$. Since district magnitudes are rank-ordered, two consecutive districts can be expressed as

$$M_{d+1} = M_d + \Delta_d \tag{3.59}$$

where $\Delta_d \geq 0$ refers to the magnitude difference between M_d and M_{d+1}.

Suppose there is a distribution of seats $\mathbf{S}_k$ that produces a total number of seats S_T. $\mathbf{S}_k$ is a $1xD$ vector that shows the number of seats that a party, $p \in \mathbf{P}$, wins in each district. In this distribution S_d seats are won in district M_d. Remember that

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) = \sum_{d=1}^{D} \frac{M_d}{M} V_{S_d}^{nec} \tag{3.60}$$

So the necessary number of votes to win S_d seats is

$$\frac{M_d}{M} \left[\frac{S_d - 1 + c}{M_d - 1 + Pc} \right] \tag{3.61}$$

Now suppose that there is another distribution of seat $\mathbf{S}_j^*$ that also produce the same total number of seats, S_T. In this new distribution, $S_d - 1$ seats are won in district M_d and 1 seat is won in district M_{d+1}. Using expression 3.59 above, the necessary number of votes to win S_d seats is

$$\frac{M_d}{M}\left[\frac{(S_d-1)-1+c}{M_d-1+Pc}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{c}{M_d+\Delta_d-1+Pc}\right] \quad (3.62)$$

To prove part b) of theorem the following must hold

$$\frac{M_d}{M}\left[\frac{S_d-1+c}{M_d-1+Pc}\right] \geq \frac{M_d}{M}\left[\frac{(S_d-1)-1+c}{M_d-1+Pc}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{c}{M_d+\Delta_d-1+Pc}\right] \quad (3.63)$$

Since all district magnitudes are the same, $\Delta_d = 0$, so expression 3.63 can be simplified as

$$S_d - 1 + c \geq S_d - 2 + 2c \quad (3.64)$$

Solving for c

$$1 \geq c \quad (3.65)$$

which is always true since $c = 0.5$.

For the second part of this proof, suppose now that there is a distribution of seats $\mathbf{S}_k$ that produces a total number of seats S_T. In this distribution $S_d - 1$ seats are allocated in district M_d and 1 seat is allocated in district M_{d+1}. The necessary number of votes to win S_d seats is

$$\frac{M_d}{M}\left[\frac{(S_d-1)-1+c}{M_d-1+Pc}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{c}{M_d+\Delta_d-1+Pc}\right] \quad (3.66)$$

Suppose now another distribution of seats, $\mathbf{S}_j^*$, that also produces S_T. In this distribution, $S_d - 2$ seats are allocated in district M_d and 2 seats are allocated in district M_{d+1}. The necessary number of votes to win S_d seats is

$$\frac{M_d}{M}\left[\frac{(S_d-2)-1+c}{M_d-1+Pc}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{1+c}{M_d+\Delta_d-1+Pc}\right] \quad (3.67)$$

If part b) of theorem 4 holds, the following must be true,

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, c, P) = V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P) \tag{3.68}$$

Since all district magnitudes are equal, $\Delta_d = 0$ and from expressions 3.66 and 3.67, expression 3.68 above can be simplified as

$$S_d - 2 + 2c = S_d - 2 + 2c \tag{3.69}$$

as stated by part b) of theorem 4.

If this proof holds for any two consecutive rank-ordered districts, then by induction it can be said that when $c = 0.5$ the minimum value of $V_{S_T}^{nec}$ is obtained in any combination of seats, $\mathbf{S}_j^*$, where $S_d > 0$ for all $S_d \in \mathbf{S}_j^*$ ■

As in the case of theorem 3, some examples may help clarify the method illustrated in theorem 4. From Table 3.2 the following even distribution of districts is offered,

$$\mathbf{M}_d = \left[\begin{array}{ccccc} 20 & 20 & 20 & 20 & 20 \end{array} \right] \tag{3.70}$$

Note, how the value of $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$ for

$$\mathbf{S}_j^* = \left[\begin{array}{ccccc} 0 & 0 & 10 & 20 & 20 \end{array} \right] \tag{3.71}$$

is the same as for

$$\mathbf{S}_j^* = \left[\begin{array}{ccccc} 10 & 10 & 10 & 10 & 10 \end{array} \right] \tag{3.72}$$

when $c = 1$. However, when $c = 0.5$ the value of $V_{S_T}^{nec}$ for 3.71 is higher than the value of $V_{S_T}^{nec}$ for 3.72. But note how the following two distributions produce the same value of $V_{S_T}^{nec}$ when $c = 0.5$.

$$\mathbf{S}_j^* = \left[\begin{array}{ccccc} 1 & 1 & 8 & 20 & 20 \end{array} \right] \tag{3.73}$$

and

$$\mathbf{S}_j^* = \left[\begin{array}{ccccc} 10 & 10 & 10 & 10 & 10 \end{array} \right] \tag{3.74}$$

The examples provided for theorems 3 and 4 help to explain the methods used to identify the combination of seats that produces $\min V_{S_T}^{nec}$ for any complete electoral system with a divisor-based electoral. Next, I consider complete electoral systems using quota-based electoral formulae.

3.2.3 Optimizing $V_{S_T}^{nec}$ for $S_T = \dfrac{M}{2}$ using quota-based electoral formulae.

As I briefly explained in the previous chapter, quota-based electoral formulae use a fixed quota of votes which is obtained using M_d. It should also be remembered that the number of full quotas achieved by each political party amounts to the number of seats that they get plus one more seat if their remainder is among the largest ones. Taking this into consideration, it is necessary to establish which is the combination of seats, $\mathbf{S}_j^*$, that produces $\min V_{S_T}^{nec}$ in complete electoral systems using quota-based methods. The following tables provide information that suggests some initial intuitions about the combinations of seat which may produce, $\mathbf{S}_j^*$.

Tables 3.5, 3.6 and 3.7 show that for quota-based electoral formula the combination of seats that produces $\min V_{S_T}^{nec}, \mathbf{S}_j^*$, is the same as for a complete electoral system with a divisor-based electoral formula, $c = 0.5$. The following theorems explain the methods used to find $\mathbf{S}_j^*$ for complete electoral systems with quota-based electoral formulae.

Theorem 5 *Given a complete electoral system with a parliament of size M, a number of districts, D, an unequal distribution of districts $\mathbf{M}_d$, a number of parties P and a quota-based electoral formula*[5]*, n, the combination of seats that produces $\min V_{S_T}^{nec}$, $\mathbf{S}_j^*$, is one in which S_T must be distributed among all districts in $\mathbf{M}_d$. Seats are distributed by first filling up the smallest districts until completed and then filling up the larger districts. However, none of*

[5] Assume that n can only take two values $n = 0$ and $n = 1$ which correspond to the Hare quota and the Droop quota respectively.

Table 3.5: Combinations of districts and seats for n, M=100 and D=5

M = 100, P = 3, D = 5, S_T= 50		
$\mathbf{M}_d$		
20 20 20 20 20	$n = 0$	$n = 1$
$\mathbf{S}_j$	$V_{S_T}^{nec}$	$V_{S_T}^{nec}$
0 0 10 20 20	0.48	0.4667
1 1 10 19 19	0.4667	0.4603
10 10 10 10 10	**0.4667**	**0.4603**
$\mathbf{M}_d$		
70 15 5 5 5		
$\mathbf{S}_j$		
50 0 0 0 0	0.4933	0.4897
35 15 0 0 0	0.4867	0.4793
20 15 5 5 5	**0.4667**	**0.4481**
M_D		
50 20 20 5 5		
$\mathbf{S}_j$		
50 0 0 0 0	0.4933	0.4869
1 19 20 5 5	**0.4667**	**0.4494**
0 20 20 5 5	0.4733	0.4524

these larger districts can have no representation. Larger districts must have at least 1 *seat each.*

Proof. Let CES stand for a complete electoral system with a parliament of size M, a number of districts, D, a distribution of districts $\mathbf{M}_d = \left[\; M_1, \;\; M_2, \;\; ..., \;\; M_D \;\right]$ restricted to $\sum_{d=1}^{D} M_d = M$, a number of parties P and a quota-based electoral formula n. Assume that $M_d < ... < M_D$ and be any two consecutive districts such that

$$M_{d+1} = M_d + \Delta_d \qquad (3.75)$$

Table 3.6: Combinations of districts and seats for n, M=100 and D=8

M = 100, P = 3, D = 8, $\mathbf{S}_T$= 50		
$\mathbf{M}_d$		
13 13 13 13 12 12 12 12	$n=0$	$n=1$
$\mathbf{S}_j$	$V_{S_T}^{nec}$	$V_{S_T}^{nec}$
13 13 13 11 0 0 0 0	0.4733	0.4519
1 1 1 1 10 12 12 12	**0.4467**	**0.4371**
0 0 0 2 12 12 12 12	0.4667	0.4462
$\mathbf{M}_d$		
50 15 15 4 4 4 4 4		
$\mathbf{S}_j$		
50 0 0 0 0 0 0 0	0.4933	0.4869
1 14 15 4 4 4 4 4	**0.4467**	**0.4188**
0 15 15 4 4 4 4 4	0.4533	0.4217
$\mathbf{M}_d$		
20 20 20 20 5 5 5 5		
$\mathbf{S}_j$		
20 20 10 0 0 0 0 0	0.48	0.467
1 1 8 20 5 5 5 5	**0.4467**	**0.4286**
0 0 10 20 5 5 5 5	0.46	0.4349

where $\Delta_d \geq 0$ refers to the magnitude difference between M_d and M_{d+1}.

Let S_T be distributed in $\mathbf{S}_k$ where S_d seats are won in district M_d. The value of $V_{S_T}^{nec}$ for this particular distribution of seats is

$$\frac{M_d}{M}\left[\frac{P(S_d-1)+n+1}{P\left(M_d+n\right)}\right] \tag{3.76}$$

Likewise, let $\mathbf{S}_j^*$ be a particular distribution of seats also producing S_T but where S_d-1 seats are won in district M_d and 1 seat

Table 3.7: Combinations of districts and seats for n, M=100 and D=20

M = 100, P = 3, D = 20, $\mathbf{S}_T$**= 50**		
$\mathbf{M}_d$		
50 5 5 5 5 5 5 5 4 1 1 1 1 1 1 1 1 1 1 1	$n = 1$	$n = 1$
$\mathbf{S}_j$	$V_{S_T}^{nec}$	$V_{S_T}^{nec}$
50 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0.4933	0.4869
1 4 5 5 5 5 5 5 4 1 1 1 1 1 1 1 1 1 1 1	**0.3667**	**0.3364**
0 5 5 5 5 5 5 5 4 1 1 1 1 1 1 1 1 1 1 1	0.3733	0.3382
$\mathbf{M}_d$		
20 20 13 12 11 3 3 2 2 2 2 2 2 1 1 1 1 1 1 1		
$\mathbf{S}_j$		
20 20 10 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0.48	0.4644
1 1 1 12 11 3 3 2 2 2 2 2 2 1 1 1 1 1 1 1	**0.3367**	**0.3466**
0 0 3 12 11 3 3 2 2 2 2 2 2 1 1 1 1 1 1 1	0.38	0.3525

is won in district M_{d+1}. The value of of $V_{S_T}^{nec}$ for $\mathbf{S}_j^*$ is

$$\frac{M_d}{M}\left[\frac{P(S_d-2)+n+1}{P\,(M_d+n)}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{n+1}{P\,(M_d+\Delta_d+n)}\right] \quad (3.77)$$

For theorem 5 to hold true the following must hold

$$V_{S_T}^{nec}(M,\mathbf{M}_d,\mathbf{S}_k,P,n) \geq V_{S_T}^{nec}(M,\mathbf{M}_d,\mathbf{S}_j^*,P,n) \quad (3.78)$$

Simplifying expressions 3.76 and 3.77 we obtain

$$\frac{M_dP}{M_d+n} \geq \frac{(M_d+\Delta_d)(n+1)}{M_d+\Delta_d+n} \quad (3.79)$$

When $n = 0$, then expression 3.79 above is reduced to

$$\frac{M_dP}{M_d} \geq \frac{M_d+\Delta_d}{M_d+\Delta_d} \quad (3.80)$$

Solving for P

$$P \geq 1 \quad (3.81)$$

which is always true since by definition $P \geq 2$

When $n = 1$ and multiplying both terms of 3.79 by $\frac{1}{2}$

$$\frac{M_d P}{2(M_d + 1)} \geq \frac{M_d + \Delta}{M_d + \Delta + 1} \tag{3.82}$$

Assume that

$$\frac{M_d + \Delta}{M_d + \Delta + 1} \approx 1 \tag{3.83}$$

then

$$\frac{M_d P}{2(M_d + 1)} \geq 1 \tag{3.84}$$

Solving for P

$$P \geq \frac{2M_d + 2}{M_d} \tag{3.85}$$

Note that in this proof $S_d > 1$. If $M_d = 1$, then $S_d - 1 = 0$ which runs against the assumption of this proof. If $M_d = 2,$ then $P > 3$ and when $M_d \geq 3$, $P < 3$. So if $M_d \geq 3$, then, no matter the number of parties and by induction,

$$V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}_k, P, n) > V^{nec}_{S_T}(M, \mathbf{M}_d, \mathbf{S}^*_j, P, n) \tag{3.86}$$

■

The following two examples will illustrate the method explained in theorem 5. The complete electoral system described in Table 3.5 shows the following distribution of districts,

$$\mathbf{M}_d = \left[\begin{array}{ccccc} 70 & 15 & 5 & 5 & 5 \end{array} \right] \tag{3.87}$$

when $n = 0$ and $n = 1$, the distribution of seats that produces $\min V^{nec}_{S_T}$, is

$$\mathbf{S}_j = \left[\begin{array}{ccccc} 20 & 15 & 5 & 5 & 5 \end{array} \right] \tag{3.88}$$

which is consistent with what theorem 5 says. In this distribution, seats are distributed according to a bottom-up criterion. In other words, seats are first allocated in the smallest districts, and allocation in the biggest district only occurs if all intermediate districts are completely filled up. In this case and only in this, the remaining seats that are still required to reach the total number of seat, S_T, 20 for $\mathbf{S}_j$ in 3.88, are allocated to the biggest district.

Table 3.7 shows an example that summarizes the basic idea of theorem 5. Let the following be the distribution of districts,

$$\mathbf{M}_d = [50\ 5\ 5\ 5\ 5\ 5\ 5\ 5\ 5\ 4\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1] \tag{3.89}$$

the combination of seats that produces $\min V_{S_T}^{nec}$, is

$$\mathbf{M}_d = [1\ 4\ 5\ 5\ 5\ 5\ 5\ 5\ 5\ 4\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1] \tag{3.90}$$

Note here how all districts must have at least one seat, but always following the bottom-up criterion.

The examples provided here refer only to those systems with districts of different sizes. The following theorem illustrates the method used to find the combination of seats that produces $\min V_{S_T}^{nec}$ when all districts have the same size.

Theorem 6 *Given a complete electoral system with a quota-based electoral formula where the number of seats in the parliament are distributed evenly, i.e.* $M_d = M_D$ *for all* $M_d \in \mathbf{M}_d$*, then, the value of* $V_{S_T}^{nec}$ *is the same for all possible combinations of seats that produce* S_T *if all seats are distributed among all districts,* $S_d^* > 0$ *for all* $S_d^* \in \mathbf{S}_j^*$*. In other words, all particular combinations of seats that do not include a seat in any of the districts produce a higher value of* $V_{S_T}^{nec}$.

Proof. Let CES stands for a complete electoral system with a parliament of size M, a number of districts, D, a distribution of districts

$\mathbf{M}_d = \left[\ M_1,\ \ M_2,\ \ ...,\ \ M_D\ \right]$ restricted to $\sum_{d=1}^{D} M_d = M$, a number of parties P and a quota-based electoral formula, n. Assume that $M_d \leq ... \leq M_D$ and be any two consecutive districts such that

$$M_{d+1} = M_d + \Delta_d \tag{3.91}$$

where $\Delta_d \geq 0$ refers to the magnitude of the difference between M_d and M_{d+1}.

Let S_T be distributed in the $1xD$ vector $\mathbf{S}_k$ where S_d seats are won in district M_d. The value of $V_{S_T}^{nec}$ for this particular distribution of seats is

$$\frac{M_d}{M}\left[\frac{P(S_d-1)+n+1}{P\left(M_d+n\right)}\right] \tag{3.92}$$

Likewise, if $\mathbf{S}_j^*$ is another particular distribution of seats also producing S_T but where $S_d - 1$ seats are won in district M_d and 1 seat is won in district M_{d+1}. The value of of $V_{S_T}^{nec}$ for $\mathbf{S}_j^*$ is

$$\frac{M_d}{M}\left[\frac{P(S_d-2)+n+1}{P\left(M_d+n\right)}\right]+\frac{M_d+\Delta_d}{M}\left[\frac{n+1}{P\left(M_d+\Delta_d+n\right)}\right] \tag{3.93}$$

According to theorem 6 the following must be true

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, P, n) \geq V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, P, n) \tag{3.94}$$

Since all district magnitudes are the same, $\Delta_d = 0$. Using expressions 3.92 and 3.93, expression 3.94 above can be reduced to

$$P(S_d-1)+n+1 \geq P(S_d-2)+2n+2 \tag{3.95}$$

Solving for P

$$P \geq n+1 \tag{3.96}$$

So, when $n = 0$

$$P \geq 1 \tag{3.97}$$

which is true by definition.

When $n = 1$

$$P \geq 2 \tag{3.98}$$

which is also true by definition since $P \geq 2$. If this holds for any two consecutive rank-ordered districts, then by induction it must also be true that that $\mathbf{S}_j^*$ produce $\min V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, c, P)$ since

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, n, P) \geq V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j^*, n, P) \tag{3.99}$$

■

Examples to illustrate this theorem can be taken from Table 3.5. The complete electoral system described there offers the following distribution of districts,

$$\mathbf{M}_d = \begin{bmatrix} 20 & 20 & 20 & 20 & 20 \end{bmatrix} \tag{3.100}$$

See the following 2 distributions of seats.

$$\mathbf{S}_j = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 \end{bmatrix} \tag{3.101}$$

and

$$\mathbf{S}_k = \begin{bmatrix} 1 & 1 & 10 & 19 & 19 \end{bmatrix} \tag{3.102}$$

As theorem 6 explains we find that those combinations of seats where all districts have at least 1 seat have the same value in $V_{S_T}^{nec}$. However, note that for the following distribution

$$\mathbf{S}_j = \begin{bmatrix} 20 & 20 & 10 & 0 & 0 \end{bmatrix} \tag{3.103}$$

this is not true and the value of $V_{S_T}^{nec}$ for this distribution of seats is higher than for 3.101 and 3.102.

3.3 Optimizing $V_{S_T}^{nec}$ for mixed electoral systems with 2 tiers and independent electoral formulae.

Put simply, mixed electoral systems are systems which employ more than one electoral formula. Normally they consist of a combination of majoritarian and proportional formulae.[6]. They usually comprise two or more tiers of seat allocation. Mixed electoral systems can be classified into two subtypes: dependent and independent mixed electoral systems (Golder 2005; Massicotte and Blais 1999; Shugart and Wattenberg 2001). Dependent mixed electoral systems are those in which the application of one electoral formula depends on the results obtained from the other formula. The classic example of this type of dependent mixed electoral system is the one used in Germany. Independent mixed electoral systems are those in which the allocation of seats in the different tiers are not connected. In other words, the outcome resulting from the application of the first electoral formula does not affect the outcome of the second formula. In the lower tier, seats are allocated in accordance with the share of votes won by each party. In the upper tier, seats are allocated without regard for the seats won by each party in the lower tier. Examples of mixed electoral systems with independent electoral formulae are those used in Russia or Ukraine for their last legislative elections.

As I will show later in Chapter 7, for the purpose of this research the only mixed electoral systems that will be taken into consideration are those with two or more independent electoral formulae. Since threshold functions are applied *ex ante* elections take place, dependent mixed electoral systems cannot be classified because the allocation in one of the tiers depends on how parties perform in the election itself. Threshold functions can be applied, though, to those electoral systems where allocation in tiers are independent.

As already mentioned, independent electoral systems normally comprise two tiers. In the case of the first electoral tier a certain number of seats are awarded in single-member districts. In the case of the second electoral tier, the remaining seats are distributed in a single district, nation-wide, and allocated using some proportional representation electoral formula. When this is the case, these

[6]A much more detailed account of the typology and functioning of these electoral systems is provided in Chapter 7.

electoral systems can be characterized according to the criterion $S_T = \frac{M}{2}$ using the following steps.

Step 1) Tier 1

Let us define M_{T1} as the number of seats that are to be distributed in Tier 1. Since it is assumed that $M_d = 1$, function $V_{S_{T1}}^{nec}$ has the following form

$$V_{S_{T1}}^{nec} = \frac{1}{PM_{T1}} S_{T1} \tag{3.104}$$

S_{T1} refers to the total number of seats that need to be won in Tier 1 in order to obtain S_T seats with the minimum number of votes.

Step 2) Tier 2

Let us define M_{T2} as the number of seats that are to be distributed in Tier 2 and S_{T2} the number of seats that must be won in Tier 2 in order to win S_T with the minimum number of votes. Then,

$$V_{S_{T2}}^{nec} = \frac{S_{T2} - 1 + c}{M_{T2} - 1 + Pc} \tag{3.105}$$

Solving for S_{T2},

$$S_{T2} = V_{S_{T2}}^{nec}(M_{T2} - 1 + Pc) + 1 - c \tag{3.106}$$

Since $V_{S_{T1}}^{nec} = V_{S_{T2}}$, the previous expression can be transformed into

$$S_{T2} = \frac{1}{PM_{T1}} S_{T1}(M_{T2} - 1 + Pc) + 1 - c \tag{3.107}$$

For $S_T = \frac{M}{2}$, the combination of seats that produces the min $V_{S_T}^{nec}$ can be found by solving the following equation,

$$\begin{cases} S_{T2} = \dfrac{1}{PM_{T1}} S_{T1}(M_{T2} - 1 + Pc) + 1 - c \\ S_{T1} + S_{T2} = \dfrac{M}{2} \end{cases} \tag{3.108}$$

Solving for S_{T1}, we obtain the following

$$S_{T1} = \text{integer} \, \frac{PM_{T1}(M + 2c - 2)}{2(M_{T1}P + M_{T2} - 1 + Pc)} \tag{3.109}$$

So, the combination of seats, S_j^*, that produces $\min V_{S_T}^{nec}$ is $S_j^* = \left[\begin{array}{cc} S_{T1} & S_{T2} \end{array} \right]$

In cases where the second tier has a quota-based electoral formula, the procedure is the following:

Step 1) Tier 1

Let us define M_{T1} as the number of seats that are to be distributed in Tier 1. Since $M_d = 1$,

$$V_{S_{T1}}^{nec} = \frac{1}{PM_{T1}} S_{T1} \tag{3.110}$$

S_{T1} refers to the total number of seats that must be won in Tier 1 in order to obtain S_T with the minimum number of votes.

Step 2) Tier 2

Let us define M_{T2} as the number of seats that are to be distributed in Tier 2 and S_{T2} as the number of seats that must be won in Tier 2 in order to win S_T with the minimum number of votes. Then,

$$V_{S_{T2}}^{nec} = \frac{P(S_{T2} - 1) + 1 + n}{P(M_{T2} + n)} \tag{3.111}$$

Since $V_{S_{T1}}^{nec} = V_{S_{T2}}$, the previous expression can be transformed to

$$S_{T2} = \frac{V_{S_{T1}}^{nec} P(M_{T2} + n) + P - 1 - n}{P} \tag{3.112}$$

which can be simplified to

$$S_{T2} = \frac{S_{T1} P(M_{T2} + n) + PM_{T1}(P - 1 - n)}{P^2 M_{T1}} \tag{3.113}$$

For $S_T = \dfrac{M}{2}$, the combination of seats that produces the min $V_{S_T}^{nec}$ can be discovered by solving the following equation,

$$\begin{cases} S_{T2} = \dfrac{S_{T1} P(M_{T2} + n) + PM_{T1}(P - 1 - n)}{P^2 M_{T1}} \\ S_{T1} + S_{T2} = \dfrac{M}{2} \end{cases} \tag{3.114}$$

Solving for S_{T1}, we obtain the following

$$S_{T1} = \text{integer}\, \frac{PM_{T1}\left[PM - 2(P - 1 - n)\right]}{2\left[P(M_{T2} + n) + P^2 M_{T1}\right]} \tag{3.115}$$

Again, the combination of seats, S_j^*, that produces min $V_{S_T}^{nec}$ is $S_j^* = \left[\begin{array}{cc} S_{T1} & S_{T2} \end{array}\right]$

The following example illustrates the application of this method. Imagine a country with a parliament of 150, $M = 150$, which is elected in two tiers. In the first tier, 100 seats are chosen in a single-member district, $M_{T1} = 100$. In the second tier, a single district with the remaining 50 seats, M_{T2}, is ruled by the divisor-based d'Hondt electoral formula, $c = 1$. The number of competing parties in this election is 4, $P = 4$. This information allows us to calculate the number of seats in both tiers that produces min $V_{S_T}^{nec}$. Using expression 3.109 above we get,

$$S_{T1} = \text{integer}\, \frac{PM_{T1}(M + 2c - 2)}{2(M_{T1}P + M_{T2} - 1 + Pc)} = 66 \tag{3.116}$$

and from here we can obtain the value for S_{T2}

$$S_{T2} = \frac{M}{2} - S_{T1} = 9 \tag{3.117}$$

So, the combination of seats that produces $\min V_{S_T}^{nec}$, S_j^*, is $\begin{bmatrix} 66 & 9 \end{bmatrix}$, which must be read as the seats that must be won in the first tier and the seats that are won in the second tier.

If tier 2 allocates its seats using a quota-based electoral formula such as the Hare formula, $n = 0$, expression 3.115 must be used. So,

$$S_{T1} = \text{integer} \frac{PM_{T1}\left[PM - 2(P-1-n)\right]}{2\left[P(M_{T2}+n) + P^2 M_{T1}\right]} = 66 \tag{3.118}$$

and the value for S_{T2}

$$S_{T2} = \frac{M}{2} - S_{T1} = 9 \tag{3.119}$$

So, the combination of seats that produces $\min V_{S_T}^{nec}$, S_j^*, is also $\begin{bmatrix} 66 & 9 \end{bmatrix}$.

Having elaborated a method to optimize aggregated threshold functions, I will now apply them to all electoral systems used between 1945 and 2000. First, however, I will present the data and methodological procedures used to perform these operations.

Chapter 4

Considerations about data and methodology

This chapter describes the data that I will use to characterize any complete electoral system. In the first section, I outline the criteria used to select the data. I also introduce the variables that will be used to calculate the values of $V_{S_T}^{nec}$. In the second section, I provide an alternative method to apply when one of the variables such as the distribution of seats between all districts, $\mathbf{M}_d$, is missing. This alternative method is based on a proxy function inspired in the original threshold function and is based on different assumptions about the combination of seats that produces $min\ V_{S_T}^{nec}$. This function must be seen simply as a practical solution to the problems posed by those cases in which this particular data is missing. Finally, I discuss the issue and importance of the legal threshold.

4.1 Data.

The data used in this research has been drawn from Golder (2005). For his research on presidential and parliamentary elections, Golder uses the Alvarez, Cheibub, Limongi and Przeworski (2000) dataset (ACLP) as a template for his own dataset. The same procedure is used here. Golder´s dataset is used as a template to build the dataset needed to characterize electoral systems. The unit of analysis for this new dataset are the electoral systems used in elections to the lower house in each country.

Three criteria have been used to select each observation. The first is a temporal one. The election must have occurred between 1945 and 2000. Secondly, I have applied a political criterion. The electoral system must have been used in a election occurred in a period of democracy. A democratic period is defined as one in which two or more political parties compete periodically for office or as Przeworski defines "democracy is a system in which parties lose elections" (Przeworski 1991;2000). Finally, an institutional criterion has been used. Only those electoral systems that allow the application of aggregated threshold functions have been chosen. As noted above, aggregated threshold functions apply a method to calculate *ex ante,* i.e. without taking into account any election results, the number of votes required to win a given number of seats. In this sense, and as I will show in more detail in the chapters that follow, some mixed electoral systems and some winner-takes-all electoral systems, for example, cannot be classified precisely because of their dependence on actual election results.

In total, 595 parliamentary elections meet these three criteria. These elections are distributed among 102 countries and involved the use of 184 different electoral systems. Table 4.1 provides the list of all countries distributed geographically, giving the total number of elections and total number of electoral systems used between 1945 and 2000.

Table 4.1: Countries by region

Region	Number of elections	Number of electoral systems
Sub-Saharan Africa		
Benin	3	3
Cape Verde	2	2
Central African Republic	2	1
Comoros	2	1
Congo	3	2
Ghana	1	1
Madagascar	2	2
Malawi	2	2
Mali	2	2
Namibia	2	1
Niger	2	1
Nigeria	4	3
Sao Tome and Principe	3	1
Sierra Leone	3	2
Somalia	2	1
Sudan	2	2
Uganda	1	1
Zambia	2	1
South Asia		
Bangladesh	3	1
India	12	2
Nepal	3	1
Pakistan	5	1
Sri Lanka	10	3
East Asia		
Japan	2	1
Mongolia	3	2
South Korea	5	3
Taiwan	1	1

Table 4.1: Countries by region (cont.)

Region	Number of elections	Number of electoral systems
South East Asia		
Indonesia	1	1
Myanmar	2	2
Philippines	8	2
Thailand	7	2
Pacific Islands-Oceania		
Kiribati	6	1
Marshall Islands	3	1
Micronesia	5	2
New Zealand	17	2
Palau	2	1
Papua New Guinnea	5	1
Solomon Islands	4	2
Middle East-North Africa		
Israel	15	3
Lebanon	7	2
Turkey	9	2
Latin America		
Argentina	14	6
Bolivia	5	3
Brazil	10	4
Chile	9	3
Colombia	10	2
Costa Rica	13	2
Dominican Republic	9	4
Ecuador	15	3
El Salvador	6	2
Guatemala	14	5
Honduras	6	1
Nicaragua	1	1
Panama	8	3

Table 4.1: Countries by region (cont.)

Region	**Number of elections**	**Number of electoral systems**
Latin America		
Peru	6	3
Uruguay	10	2
Venezuela	2	2
Non-hispanic America		
Antigua	4	1
The Bahamas	5	1
Barbados	8	1
Belize	4	1
Canada	17	1
Dominica	5	1
Grenada	5	1
Guyana	2	1
Haiti	2	1
Jamaica	9	2
St. Kitts and Nevis	5	1
St. Lucia	6	1
St. Vincent	5	2
Surinam	4	1
Trinidad and Tobago	8	1
United States	28	1
Eastern Europe		
Albania	2	2
Armenia	2	2
Bulgaria	4	2
Croatia	3	3
Kyrgyztan	2	2
Latvia	3	1
Lithuania	3	1
Macedonia	2	2
Moldova	1	1

Table 4.1: Countries by region (cont.)

Region	Number of elections	Number of electoral systems
Eastern Europe		
Poland	3	2
Russia	3	1
Slovak Republic	1	1
Ukraine	2	2
Western Europe		
Andorra	2	1
Finland	15	2
France	12	4
Greece	2	2
Iceland	5	1
Liechtenstein	2	1
Luxemburg	10	4
Netherlands	16	2
Norway	10	3
Portugal	9	2
San Marino	2	1
Spain	8	1
Sweden	7	2
Switzerland	14	2
United Kingdom	14	1
Total	595	184

By electoral system I mean a complete electoral system. It will be remembered from the previous chapter that a complete electoral system comprises the following elements: an electoral formula, c or n, the number of districts in which the country is divided, D, a $1xD$ vector with all district magnitudes, $\mathbf{M}_d$, the number of seats in the legislative assembly, M, and finally the number of parties competing in all districts, P. In accordance with this definition the criteria that have been used to differentiate between two different complete electoral systems are:

a) A change of the electoral formula, c or n. This may also include a modification in the number of tiers used to allocate seats.

b) A change in the number of districts into which the country is divided, D.

c) A change in the number of seats in the legislative assembly. A change is considered to have taken place when there has been a change of 20% in the size of M (Lijphart 1994; Golder 2005).

d) Finally, it should also be noted that the electoral system established after a period of dictatorship is also considered to be a new electoral system.

The following examples illustrate how these criteria have been applied:

a) The systems used in the elections in Israel in 1969 and in 1973 are treated as different electoral systems because in 1969 a quota-based electoral formula was used, whereas in the 1973 election, a divisor-based electoral formula was implemented. In El Salvador, the 1988 election adopted an electoral system based on 1 tier of seat allocation whereas in the 1991 election a 2 tier of seat allocation system was employed.

b) The electoral system used in the 1995 elections in Benin is considered to be different to that used in the 1999 election because the number of districts was different; whereas in the 1995 election the seats were distributed among 18 districts, in the 1999 election this number rose to 24.

c) In the 1958 election in Brazil 326 members were elected to parliament. In the following election in 1962 that number increased to 404 members. This change in the size of parliament is treated here as constituting a change in the electoral system.

d) Even though the electoral system used in the 1971 election in Uruguay was the same as that used in the 1989 election, these 2 electoral systems are treated as different because of the dictatorship that existed between those two elections.

When more than one electoral system has been used in a country in the period under consideration, this is indicated in the data by the use of ordinals. So Argentina1 and Argentina2 refer to 2 different complete electoral systems that have been used in the same country. Table 4.2 shows the total number of elections and electoral systems found for each of the different types of electoral system. This table shows that *winner-takes-all* electoral systems and proportional representation electoral systems have been used for about the same number of elections. Winner-takes-all electoral systems have been used for 264 elections whereas 261 elections occurred under proportional representation electoral systems. The number of countries using these electoral systems is also about the same. More specifically, 50 countries have used majoritarian methods whereas 47 have used proportional representation during the period between 1945 and 2000. Multi-tier and mixed electoral systems have been used in 6 and 23 countries respectively.

Table 4.2: Number of elections and electoral systems

Type of Electoral System	**Elections**	**Electoral Systems**
Winner-takes-all	264	69
P.R.-Divisors	170	45
P.R.-Quota	91	34
Multi-Tier	25	9
Mixed	45	27
Total	595	184

A number of further conclusions can be drawn from Table 4.2. It suggests, for example, some conclusions about the stability of electoral systems. *Winner-takes-all* electoral systems show greater institutional stability, as can be seen from the fact that the ratio between the total number of these electoral systems and the total elections held under them is just 0.26. Proportional representation show greater instability,as this ratio is 0.30. However, it can be seen that divisor-based electoral systems, with a ratio of 0.26, produce more stable electoral systems than quota-based, 0.37. Greater instability appears to characterize multi-tier and mixed electoral systems. The ratio for multi-tier electoral systems is 0.36, a figure which rises to 0.60 in the case of mixed-electoral systems.

Finally, a few words about the number of parties, P. As I noted in previous chapters, the total number of competing parties is assumed to be the same in all the districts. However, for the purposes of this research I use the effective number of parties as defined by Taagepera and Shugart instead of the total number of competing parties. The effective number of parties is defined as the "number of hypothetical equal-sized parties that would have the same effect on fractionalization of the party system as have the actual parties of varying sizes."(Taagepera and Shugart 1989:79). The following examples will illustrate this idea more clearly.

Imagine a complete electoral system where there are 4 parties, $\mathbf{P} = \begin{bmatrix} A & B & C & D \end{bmatrix}$. Suppose the following distribution of votes in all districts for these between these parties, $\mathbf{V}_d$.

Table 4.3: Effective Number of Parties

	Parties			
$\mathbf{V}_d$	A	B	C	D
1	0.48	0.47	0.04	0.01
2	0.46	0.27	0.22	0.05
3	0.34	0.33	0.18	0.15
4	0.25	0.25	0.25	0.25

Although the number of parties is the same in all the distributions, it can be seen that in distribution 1 there are two large parties that take practically all the votes cast, 95%, while in distribution 4 there are four parties of equal strength. Can we maintain that the number of parties in both distributions is effectively the same? Taagepera and Shugart suggest that we should take each party's electoral strength into account. This is the idea behind the concept of effective number of parties which is operationalised as follows,

$$P = \frac{1}{\sum \left(V_T^p\right)^2} \tag{4.1}$$

where V_T^p is the share of total votes won by party p.

Applying this formula, it can be seen that the number of effective parties in distribution 1 is 2.15 which is quite consistent with the bipartidism that results from the distribution of votes among all parties. Compare this distribution of votes with distributions 3 or 4. The distribution of votes 3 produces an effective number of parties of 3.57 which indicates that all votes are distributed among approximately 3 similar sized parties. Finally, in the distribution of votes 4 the effective number of parties is 4, since all parties have exactly the same share of the vote.

In this research the total number of competing parties, P, will refer to the effective number of parties. I am aware that these are two distinct concepts. As shown here, the effective number of parties is in fact calculated on the basis of a given distribution of the vote among all parties. That might be problematic when incorporated into the aggregated threshold functions since that distribution of votes may be different to the distribution of the vote assumed here when applying the functions. Strictly speaking, if the distribution of the vote assumed when applying the aggregated functions were taken into account, then the effective number of parties would have a different value. However, the use of the effective number of parties as defined here can be defended on the grounds that it constitutes the best possible proxy for the number of parties that have a chance of winning representation in each district. In other words, it is the best approximation to the number of parties actually competing for the seats at stake. Moreover, the use of this measure makes it possible to reflect the importance of the real weight of political parties. Another argument that could be made against the use of the effective number of parties is that the real weight of a political party cannot be known until after the election has taken place. This is a very reasonable objection and could undermine the emphasis that this research project places on developing a measure that can be applied without taking actual election results into account. While this may be true, it is also the case that the effective number of parties can be estimated approximately before elections actually take place. Polls and surveys provide useful information that give us perfectly valid estimates of the number of effective parties that can be used to calculate aggregated threshold functions.

4.2 Methodology.

4.2.1 Procedures used in cases with complete data.

In Chapter 2, I outlined a method that can be used to calculate the necessary number of votes to win any number of seats. I also offered some theorems that demonstrated how to optimize this method in order to find out the value that will be used to characterize any complete electoral system. This value, as I also explained there, is obtained by applying the function to calculate the necessary number of votes to a particular combination of seats. In this section I offer two examples that I hope will clarify the procedure used to classify an electoral system. The examples used in this sub-section contain all the information required to apply the aggregated threshold function with complete data. In the next sub-section, I will outline a different procedure for those cases in which all the data is not available.

Let us consider first cases in which complete data is available. Take, for example, the electoral system used in Bolivia for the 1993 elections. In these elections, 130 deputies were elected from 9 dis-

Table 4.4: Distributions of districts for the 1993 elections in Bolivia

Districts	M_d
La Paz	28
Potosí	19
Cochabamba	18
Santa Cruz	17
Chuquisaca	13
Oruro	10
Beni	9
Tarija	9
Pando	7
Total	130

tricts, and a divisor-based electoral formula was used. More specifically, the formula used was the Sainte-Lagüe formula ($c = 0.5$) and the number of parties competing in these 9 districts was $P = 4.66$. Table 4.4 shows the districts into which the country was divided.

The vector containing the distribution of districts can be obtained from Table 4.4 . This vector is as follows:

$$\mathbf{M}_d = [28\ 19\ 18\ 17\ 13\ 10\ 9\ 9\ 7] \tag{4.2}$$

According to theorem 3 for $c = 0.5$ the combination of seats that produces $\min V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ is one in which the total number of seats, S_T, must be distributed among all districts in vector $\mathbf{M}_d$. Seats are distributed first in the smallest districts. Once all the seats in the smallest districts have been assigned, the seats in the larger districts are then distributed. However, none of these larger districts can be left without representation. Larger districts must have at least 1 seat each. In the 1993 elections in Bolivia, $S_T = 65$ and following this theorem the vector of seats producing $\min V_{S_T}^{nec}$ would be,

$$\mathbf{S}_d = [1\ 1\ 1\ 14\ 13\ 10\ 9\ 9\ 7] \tag{4.3}$$

Vector $\mathbf{S}_d$ indicates the number of seats that must be won in each district in order to obtain $\min V_{S_T}^{nec}$ Hence, in La Paz, the largest district, just 1 seat must be won but in Pando, where 7 seats are contested, all of them must be won. Vectors $\mathbf{M}_d$ and consequently $\mathbf{S}_d$ are, therefore, two vital pieces of information to correctly apply the aggregated threshold function in order to calculate the necessary number of votes to win S_T seats. It will be remembered that,

$$V_{S_T}^{nec} = \sum_{d=1}^{D} \frac{M_d\,(S_d - 1 + c)}{M\,(M_d - 1 + Pc)} \tag{4.4}$$

Applying the rest of the variables, the following result is obtained,

$$V_{S_T}^{nec} = 0.4148 \tag{4.5}$$

A share of 41.48% of the vote is the minimum number of votes that was necessary to win $S_T = 65$ in the 1993 election in Bolivia. The example provided here refers to a divisor-based electoral formula. When a quota-based electoral formula is used, the method used follows the same procedure as the one shown here.

4.2.2 Procedure used in cases with missing data.

In this sub-section, I outline the method used to calculate aggregated thresholds when district data is missing. The methodology explained here is based on some assumptions that may contradict some of the theorems proposed in the previous chapter. However, I will also explain the justification for taking this approach, as well as show the strong positive correlations between the aggregated threshold values estimated with $V_{S_T}^{nec}$ and the values obtained by using a proxy function. These arguments and findings provide the ultimate justification for using this method in cases of missing data.

Four types of electoral systems have been mentioned so far: *winner-takes-all*, proportional representation, divisor-based and quota-based, multi-tier and mixed electoral systems. The functioning and the particularities of each of these types of electoral system are explained in the chapters that follow. As I showed above, when applying aggregated threshold functions it is necessary to have information about the number of districts, D, as well as about the distribution of seats among these districts, $\mathbf{M}_d$. This information is particularly important in plurinominal districts. When information about the distribution of seats among all districts is missing, aggregated threshold values for divisor-based electoral systems[1] can be calculated using the following proxy function

$$\widehat{V_{S_T}^{nec}} = \frac{\widehat{S} - 1 + c}{\widehat{M} - 1 + Pc} \tag{4.6}$$

where $\widehat{M}$ and $\widehat{S}$ refers to the average district magnitude and average number of seats respectively.

Average district magnitude is a common measure used in electoral studies. It is calculated by dividing the total number of seats by the total number of districts. Formally,

$$\widehat{M} = \frac{M}{D} \tag{4.7}$$

The average seats is a measure that standardizes the number of seats in accordance with the value of $\widehat{M}$. It is calculated as follows,

$$\widehat{S} = \frac{\widehat{M} S_T}{M} \tag{4.8}$$

So, if $S_T = \frac{M}{2}$ from expressions 4.6, 4.7 and 4.8 we obtain the following,

$$\widehat{V_{S_T}^{nec}} = \frac{M + 2D(c-1)}{2(M + D(Pc-1))} \tag{4.9}$$

When $S_T = 1$, then expression 4.6 is as follows

[1] As in the case of systems with complete data, the example provided here refers to a complete electoral system with a divisor-based electoral formula. However, the same logic and procedure must be followed when the electoral formula is based on a quota.

$$\widehat{V_{S_T}^{nec}} = \frac{1 - D(c-1)}{M + D(Pc-1)} \tag{4.10}$$

The use of this approach in cases where the district data is missing obliges us to think again about the distributions of seats that produce the minimum value of $V_{S_T}^{nec}$. As shown in the theorems above, in most cases these distributions were based on S_T seats distributed in the smaller districts. This is not the case when $\widehat{V_{S_T}^{nec}}$ is used, since this new function assumes the equal distribution of seats among all districts. This is so because all the information about the distribution of district is summarized in the average district magnitude, which is a single value and the seats won constitute a determined share of this value.

In order to illustrate this idea, let us assume that we do not have data on the distribution of districts, $\mathbf{M}_d$, for the 1993 election in Bolivia. In accordance with the procedure followed in cases of missing data, the average magnitude must be calculated in order to apply this proxy function. In the case of this example, the value of this magnitude is,

$$\widehat{M} = 14.44 \tag{4.11}$$

Consequently, the standardized number of seats is,

$$\widehat{S} = 7.22 \tag{4.12}$$

On the basis of this result we can apply the expression 4.6, which gives us the following result:

$$\widehat{V_{S_T}^{nec}} = \frac{\widehat{S} - 1 + c}{\widehat{M} - 1 + Pc} = 0.4261 \tag{4.13}$$

This method for handling cases with missing data produces logical distortions between the values obtained using aggregated threshold functions and the proxy function. As Table 4.5 shows for quota-based electoral systems, the distortion produced is 0.007, an average difference of about 0.7% between the value obtained using the aggregated threshold function and the proxy function. For divisor-based electoral systems, the data shown in Table 4.6 reveals that the average distortion between the value using aggregated threshold function and the proxy function is 0.02 which is a difference of 2% in both values.

Table 4.5: Missing data results for quota-based electoral systems

Country	**Year**	$\mathbf{V}_{S_T}^{nec}$	$\widehat{V_{S_T}^{nec}}$
Benin	1995	0.3786	0.3012
Benin	1999	0.3715	0.2873
Brazil	1947	0.4445	0.4445
Brazil	1998	0.4548	0.4538
Colombia	1974	0.4249	0.4244
Colombia	1978	0.4299	0.4295
Israel	1955	0.493	0.4929
Israel	1959	0.4933	0.4932
Luxembourg	1989	0.4248	0.4331
Luxembourg	1994	0.4248	0.4330

District data is missing for 17 of the countries in the dataset used for this analysis. This means that a total of 38 elections cannot be characterized using aggregated threshold functions in accordance with the theorems shown above. These countries are listed in Table 4.7

Table 4.6: Missing data results for divisor-based electoral systems

Country	**Year**	$\mathbf{V}_{S_T}^{nec}$	$\widehat{V_{S_T}^{nec}}$
Argentina	1983	0.3993	0.4332
Argentina	1985	0.2549	0.3100
Bolivia	1993	0.4148	0.4261
Bulgaria	1991	0.3331	0.3563
Bulgaria	1994	0.343	0.3668
Chile	1961	0.2101	0.2455
Finland	1948	0.3614	0.3866
Finland	1951	0.3674	0.3852
Spain	1982	0.3436	0.3776
Spain	1986	0.3237	0.3604

Table 4.7: Countries with missing district data

Country	**Election Year**	**Number of Districts**	**Average District**
Argentina	1963	23	8.35
Argentina	1965	23	4.17
Argentina	1973	24	10.13
Benin	1991	6	10.67
Cape Verde	1991	25	3.16
Cape Verde	1995	19	3.79
Chile	1949	28	5.25
Chile	1953	28	5.25
Chile	1957	28	5.25
Chile	1965	29	5.25
Chile	1969	60	5.36
Chile	1973	60	5.24
France	1946	102	5.33
France	1986	96	5.79
Guatemala	1966	22	2.50
Guatemala	1970	22	2.50

Table 4.7: Countries with missing district data (cont.)

Country	**Election Year**	**Number of Districts**	**Average District**
Indonesia	1999	27	17.11
Liechtenstein	1993	2	12.5
Liechtenstein	1997	2	12.5
Panama	1960	10	5.3
Peru	1962	24	7.58
Peru	1963	24	5.79
Portugal	1976	22	12.95
San Marino	1993	10	6
San Marino	1998	10	6
Sao Tome and Principe	1991	7	7.86
Sao Tome and Principe	1994	7	7.86
Sao Tome and Principe	1998	7	7.86
Somalia	1964	47	2.62
Turkey	1961	67	6.72
Turkey	1969	67	6.72
Turkey	1973	67	6.72
Turkey	1977	67	6.72
Turkey	1983	83	5.42
Turkey	1999	84	6.55
Venezuela	1947	23	4.78
Venezuela	2000	24	6.88

In order to test the predictive capacity of this proxy, I compared the results obtained by applying the aggregated threshold functions in those cases where all data is available with the results obtained in the same cases but using the proxy function above. The following table shows these results for $S_T = \frac{M}{2}$ and $S_T = 1$.

Table 4.8 shows how both the aggregated threshold function and the proxy functions works very well for $S_T = \frac{M}{2}$. The correlation is 0.967. The same cannot be said for $S_T = 1$. In this case, the correlation is very low, $r = 0.211$. The reasons for this difference lie

Table 4.8: Correlation values for aggregated threshold values and their proxy

	$\widehat{V^{nec}_{S_T=\frac{M}{2}}}$	$\widehat{V^{nec}_{S_T=1}}$
$V^{nec}_{S_T=\frac{M}{2}}$	0.967	$N.A.$
$V^{nec}_{S_T=1}$	$N.A.$	0.211

in the structure of the function itself and, above all, the value of c. When $c = 0.5$ (Sainte-Lagüe formula) very low results are obtained. In any event, the values of both $V^{nec}_{S_T=1}$ and $\widehat{V^{nec}_{S_T=1}}$ constitute special cases that will be considered in much more detail in the following section. These values represent the institutionally-determined conditions to win a single seat, but legislators may also introduce legal barriers to prevent the entry of minor parties into parliament. As, I will show later, when this is the case the values of $V^{nec}_{S_T=1}$ or $\widehat{V^{nec}_{S_T=1}}$ may not be of interest. To sum up, when district data is missing a proxy function will be used to calculate the minimum number of votes required to win $S_T = \dfrac{M}{2}$ seats. However, when $S_T = 1$, then, aggregated threshold functions will only be used for those cases in which data is available.

4.3 The importance of legal thresholds.

One of the criteria used to classify electoral systems is the minimum number of votes needed to win a single seat in the parliament in question. This measure can be of great importance for testing the degree of accessibility of the electoral system to small parties. The lower the minimum number of votes required to win one seat, the more permissive the electoral system and the greater the number of small parties that may be present in the parliament. On the contrary, if the barrier to winning a single seat is high, then the electoral system is more exclusive and therefore the presence of small parties in the parliament is rare.

The presence of a large number of political parties in parliament may be associated with the idea of the broad representation of citizens' political preferences. However, political instability may also result from the existence of too many parties with a voice in the decision-making process. As Lupia and Strøm affirm, "cabinet coalitions in multiparty parliamentary democracies lead a precarious existence" (Lupia and Strøm 1995: 648). When the parliament is highly fragmented, so too normally is the government. The stability of the cabinet depends on the support of its partners, and discrepancies between these may end in parliament being dissolved and early elections (King et al.1990). It is for this reason that many electoral systems establish a hurdle that every party must get over in order to win a seat. These mechanisms are adopted when electoral systems make it possible to win seats with relatively few votes. In such cases, but not only these, we often find that legal thresholds are established. The minimum number of votes necessary to win 1 seat calculated using the aggregated threshold function may, therefore, be meaningless when legal thresholds of this type exist. If the legal threshold is higher than the value predicted by the aggregated threshold function, then any party must win at least the proportion of votes established by that legal threshold in order to win at least 1 seat.

Table 4.9: Legal thresholds and aggregated threshold values for 1 seat in 8 democracies at national level

Country	**Year**	**Tier**	$\mathbf{T}_L$ (%)	$\mathbf{V}^{nec}_{S_T=1}$(%)
Bulgaria	1991	National	4%	0.23%
Bulgaria	1994	National	4%	0.24%
Bulgaria	1997	National	4%	0.27%
Israel	1977	National	1%	0.8%
Israel	1988	National	1%	0.8%
Moldova	1998	National	4%	0.9%
Netherlands	1998	National	1%	0.9%
Netherlands	1946	National	0.67%	0.64%

Tables 4.9 and 4.10 illustrate the contrast between the institutional response resulting from the electoral system and the barrier the legal threshold constitutes to obtaining political representation. All the electoral systems shown in these tables are P.R. electoral systems with one tier of seat allocation, though the same reasoning can be extended to other types of electoral system. Table 4.9 refers to those legal thresholds that apply at national level. In these cases, a political party must obtain the minimum proportion of the vote nationwide established by this threshold in order to win political representation. By way of example, Table 4.9 shows 8 democracies in which national legal thresholds apply. The difference between the aggregated threshold value and the legal threshold is quite considerable in the case of Bulgaria and less dramatic in the cases of Israel, Moldova and The Netherlands. One reason for these differences is that whereas Bulgaria has 31 districts the other countries have a single district. In accordance with theorem 1 above, the minimum number of votes necessary to win 1 seat is obtained when the seat is won in the smallest district. In the case of Bulgaria, the magnitude of this district is 4. In contrast, in the case of Israel this magnitude is 150, in that of Moldova it is 104 and in The Netherlands it is 100 in the 1946 election and 150 in the 1998 election. District magnitude and legal threshold are, therefore connected. At the national level, the legal threshold is really effective when the magnitude of the district is very small as in the case of Bulgaria. However, if the magnitude of the district is as big as in the case of The Netherlands in 1998, where 150 deputies were elected in a single district, then the legal threshold established and the aggregated threshold value necessary to win 1 seat is very close.

Table 4.10 shows those cases where legal thresholds apply at district level. In these cases a party must win at least the minimum percentage of the vote established by this threshold in at least 1 district in order to win political representation. Two countries of the sample have district-level legal thresholds: Spain and Latvia. These cases are very revealing in that they show the very different impact of the legal barrier. In the case of Spain, the minimum

Table 4.10: Legal thresholds and aggregated threshold values for 1 seat in 5 democracies at district level

Country	**Year**	**Tier**	$\mathbf{T}_L$ (%)	$\mathbf{V}^{nec}_{S_d=1}$(%)
Latvia	1993	District	4%	2.2%
Latvia	1995	District	4%	2.1%
Latvia	1998	District	4%	2.2%
Spain	1977	District	3%	23.31%
Spain	2000	District	3%	33.44%

number of votes necessary to win 1 seat is found in the smallest districts, namely Ceuta and Melilla, which are both uninominal districts. The threshold value necessary to win 1 seat in these two districts were 23.31% in the 1977 election and 33.44% in the 2000 election. The legal threshold in Spain is 3%. In Latvia, the situation is very different. The territory is divided into 5 districts of equal size, each of which consists of 20 seats and has a legal threshold of 4%. The threshold values necessary to win 1 seat were 2.2%, 2.1% and 2.2% for the 1993, 1995 and 1998 elections respectively. Whereas in the Spanish case the legal threshold has no effect when the magnitude of the district is so small, in the Latvian case the legal threshold does in practice establish the minimum number of votes that a party must obtain in order to win political representation. In fact, as Ramirez has noted, the effect of the legal threshold in the Spanish electoral system is only effective in the largest districts, such as Madrid and Barcelona which both send around 30 deputies to Congress. (Ramirez 1997; see also Taagepera and Shugart 1989:133).

The main conclusion of this discussion is that aggregated threshold values for $S_T = 1$ must be treated with caution precisely because they do not take the effect of legal thresholds into account. Aggregated threshold values are only relevant insofar as they are lower than the value of the legal threshold. Otherwise, the share of votes established by the legal barrier will be the determinant value to find out the point at which the first seat can be won.

Up to this point, I have been describing a method to characterize electoral systems. In the first chapter, I introduced the idea of aggregated threshold functions. In the Chapter 2, I introduced 6 theorems that optimize these functions. Finally, I have presented the data required to apply this function as well as some examples that show the way in which electoral systems will be classified.

The remaining chapters will deal strictly with the task of characterizing all the electoral systems in the dataset. I will begin with *winner-takes-all* electoral systems, then I will continue with divisor- and quota-based P.R. electoral systems. Finally, multi-tier and mixed member electoral systems will be characterized.

Chapter 5

Characterizing *Winner-takes-all* Electoral Systems

In the following three chapters I apply the method outlined in Chapter 4 to characterize electoral systems. The main purpose of the following pages is to execute the method described above, and thereby locate each electoral system on a continuum. This continuum will show the points at which each complete electoral system allows a party to obtain a parliamentary majority, on the one hand, and the minimum representation in that chamber, on the other. In this chapter attention will be paid to *winner-takes-all* electoral systems. In order to locate this analysis, I will first describe the different types of electoral formulae, namely plurality and majority, that form part of the *winner-takes-all* family of electoral systems. The difference between these two types of winner-takes-all systems lies in whether a seat is won with a plurality of the votes or with a majority of them. Once this family of electoral systems has been discussed, I will explain which of them can be characterized using aggregated threshold functions. Finally, I will introduce aggregated threshold values for countries that have used *winner-takes-all* electoral systems between 1945-2000.

5.1 Types of *winner-takes-all* electoral systems

Two different types of electoral formulae can be distinguished within *winner-takes-all* electoral systems. On the one hand, there are electoral systems based on a majoritarian electoral formula. On the other hand, there are those electoral systems based on a plurality electoral formula. The main difference between these two types of system lies in the way in which seats are won. Under the plurality electoral formula, seats are won when a party obtains a plurality of the vote. This means that the winner must obtain more votes, no matter the number, than any of their competitors. On the contrary, under a majoritarian electoral formula the winner must obtain the majority of the votes cast. That means that the winner must win at least 50% of the vote (Reynolds and Reilly 1997; Farrell 2001).

The paradigmatic example of an electoral system with a plurality electoral formula is the system known as *First-Past-the-Post* (FPTP). This is also the most simple and straightforward type of electoral system. In these, the country is usually divided into as many constituencies as seats there are in the Parliament. Hence each constituency or district is uninominal. The winner of the seat in any given constituency is the party or candidate that wins the largest number votes in that district. To give an extreme example, imagine there are Voters-1 parties competing in a single member district. Under this electoral system a seat could be won with 2 votes if all other parties obtained just 1 vote. FPTP is the most widely used *winner-takes-all* electoral system in the world. Between 1945 and 2000, 39 countries adopted FPTP and a total of 232 elections were held under this electoral system. The countries using FPTP include the United Kingdom, the United States of America and Canada. In general, most of the former British colonies adopted this electoral formula after they won independence.

There are two variations to FPTP. They are the Block Vote (BV) and the Party Block (PB) formulae. They work in exactly the same way as FPTP but they are applied to multimember as opposed to single member districts. Under the Block Vote formula, voters have as many votes as seats to be filled and they can use them freely to vote for individual candidates regardless of party affiliation. The electoral systems used in Laos in 1955 or in Mongolia in 1992 applied the BV electoral formula. The Party Block (PB) formula works in a very similar fashion to FPTP. In this, voters must choose between party lists of candidates rather then individuals. The party that obtains a plurality of the votes wins all the seats in the district and its entire list of candidates is elected. As Reynolds and Reilly points out (1997) the PB electoral formula produces a "super-majoritarian" electoral system. Lebanon used this PB electoral formula for its general elections between 1953-1974.

There are two different types of majoritarian formula: the Alternative Vote (AV) and the Two Round Systems (TRS). Electoral systems that use the AV majoritarian formula are also based on single member districts (SMDs). The main difference with FPTP lies in the way in which voters actually cast their vote. Whereas the FPTP ballot is cardinal, the AV ballot is ordinal. This means that under the AV system, rather than indicating their preferred candidate, voters rank the candidates in order of preference. For example, if 4 parties compete in a constituency, voters mark with a "1" their first preference party, they mark with "2" their second preference, and so on until the total number of competing parties are marked. Candidates obtaining over 50% of the vote are automatically elected. If none of the candidates wins that share of the vote, then the candidate with the lowest number of first preference ticks on the ballot paper is eliminated and his or her ballots are examined for their second preference. The process is repeated until one of the candidates obtains the majority of the votes. The distinctive feature of the Alternative Vote formula is that it enables voters to express their preferences rather than simply express their first choice. It is for this reason that this formula is also known as Preferential Voting (PV). Australia is the country that has used this electoral formula most continuously.

The Two Round System is the second type of majoritarian formula that is applied in SMDs. This type of formula is also known as the Run-off or Double-ballot. It is a very simple procedure. In the first of the two rounds of voting, if a candidate wins over 50% of the vote then, he or she is automatically elected. If none of the candidates wins that share of the vote, then a second round of voting takes place and the candidate who obtains a plurality of the vote is elected. Eleven countries have used this electoral formula between 1945 and 2000. France is probably the place where this method has been applied most extensively, albeit in presidential rather than parliamentary elections (Golder 2005).

Scholars do not coincide when it comes to classifying three other electoral formulae as plurality-majority or proportional representation electoral formulae. The electoral formulae in question are rarely used and hence will not receive much attention here. According to Reynolds and Reilly (1997) the Limited Vote (LV), the Cumulative Vote (CV) and the Single Non-Transferable Vote (SNTV) electoral formulae are semi-proportional methods. Others scholars such as Golder (2005) and Farrell (2001) consider that these methods in fact belong to the majority-plurality family. The common feature these three types of electoral formulae share is that they are applied to multi-member districts. Under the Limited Vote system voters have fewer votes than the number of seats to be filled. The most direct consequence of this is that it reduces the probability that a large party will have its full list of candidates elected. The Limited Vote is used in elections to the Spanish Senate and it was also used in the general elections in Argentina between 1946-1950 and then again between 1958 and 1962. The Cumulative Vote formula gives voters the same number of votes as there are seats to be filled. The voter can cast one or more votes for one of the candidates in the list, in other words, the vote can be cumulative. This method was continuously used for the Illinois House of Representatives from 1870-1980 (Farrel 2001: 46). Finally, the Single Non-Transferable

Vote (SNTV) formula gives the voter just 1 vote in multi-seats districts and those candidates winning the highest share of total votes are elected to the available seats. This method was used most notably in Japan between 1947 and 1995. Table 5.1 summarizes the classification of the different types of electoral formulae that can be found within the plurality and majority families in accordance with district size.

Table 5.1: Winner-takes-all electoral systems

	SMD	Multi-member
Plurality	First Past The Post	Block Vote Party Block
Majority	Alternative Vote Two Round System	
Miscellaneous		Limited Vote Cumulative Vote Single Non Transferable Vote

Aggregated threshold functions will only be applied to four electoral formulae: First-Past- The-Post (FPTP), Block Vote (BV), Party Block (BV) and Two Round Systems (TRS). One fundamental consideration justifies this decision. This concerns the way in which the Alternative Vote, the Limited Vote, the Cumulative Vote and the SNTV electoral formulae proceed to assign seats. As I explained above, in most cases these formulae involve a number of different steps in order to determine who wins the seat and this result is difficult to know before the election results are known. It is quite hard to determine the allocation of seats under these formulae unless the exact distribution of votes between all the candidates is assumed. This is not the case for FPTP, BV, PV, for which regardless of the distribution of the vote it is possible to calculate a minimum share of votes required to win a seat. It should be noted however that in the case of the Two Round System (TRS), aggregated threshold functions are applied to the second round only.

In general, it can be said that aggregated threshold functions can be applied to those *winner-takes-all* electoral systems that fit the following definition:

$$F(\mathbf{V}^P, M_d) = \left\{ S_d \mid S_d^p = M_d \wedge S_d^{\neg p} = 0 \Longrightarrow V_d^p > V_d^{\neg p} \textit{ and } \sum_{p=1}^{P} S_d = M_d \right\} \tag{5.1}$$

for all $M_d \in \mathbf{M}_d$

$F(\mathbf{V}^P, M_d)$ refers to a function that relates the number of votes obtained by each party to the number of seats that the party in question wins. $\mathbf{V}^P$ is $1xP$ vector that shows the share of votes won by each party. One condition of $\mathbf{V}^P$ is

$$\sum_{p=1}^{P} V^p \leq 1 \text{ for all } V^p \in \mathbf{V}^P \tag{5.2}$$

A total of 327 democratic elections held in 54 countries in the period 1945-2000 took place under one of the *winner-takes-all* electoral systems described above. Of these 327 elections, 264 were carried out under the First-Past-The-Post (FPTP), Block Vote (BV), Party Block (BV) or the Two Rounds Systems (TRS). This sample accounts for 80% of all these elections and represents 92% of the countries, since these formulae are used in 50 countries. Hence the values obtained from the application of the aggregated threshold functions cover a very large proportion of the *winner-takes-all* electoral systems used during the period under study.

5.2 Data.

The aggregated threshold function applied to those electoral systems that fit definition 5.1 above is as follows :

$$V_{S_T}^{nec} = \frac{M_d}{M} \left(\frac{S_d - 1 + c}{M_d - 1 + Pc} \right) * S_T \tag{5.3}$$

When $M_d = S_d = 1$ and $S_T = 1$, then

$$V^{nec}_{S_T=1} = \frac{1}{MP} \tag{5.4}$$

This function shows how the aggregated value required to obtain 1 seat in a *winner-takes-all* electoral system is expected to be low since it only depends on the interaction between the size of the parliament and the number of political parties. The value obtained does not necessarily mean that a party obtaining this proportion of the vote will in fact win representation. The meaning of this value is quite different: it refers merely to the minimum proportion of votes required to win a seat. The low values obtained when applying function 5.4 cannot actually be understood as indicating how permissive these electoral systems are with respect to minority parties. In real life, *winner-takes-all* electoral systems produce the most favorable results for large parties in each district since only the most-voted party wins the seat at stake. Small parties only have a chance of winning a seat when they obtain the majority or the plurality of the votes in the district where they present candidates. For this reason, smaller, nationwide parties have very limited chances of winning seats on the basis of their aggregated share of the vote. However, small regional parties are more likely to win a seat in districts in a particular region if their support in the region is widely spread.

The following table shows the results obtained when we apply the aggregated threshold function given above when $S_T = 1$. In accordance with the procedures introduced in Chapter 3, these values are calculated assuming that a party, p, wins 1 seat in 1 district with the necessary number of votes and obtain no votes in the remaining districts. For those cases in which $M_d > 1$, a similar procedure has been used taking into account the size of M_d.

Table 5.2: Aggregated threshold values for $V^{nec}_{S_T=1}$

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=1}$
Antigua1	1984	FPTP	1	1.92	17	0.0306
Antigua1	1989	FPTP	1	1.98	17	0.0297
Antigua1	1994	FPTP	1	2.07	17	0.0284
Antigua1	1999	FPTP	1	2.09	17	0.0281
Argentina1	1951	Plurality	11	1.97	158	0.0353
Argentina2	1954	Plurality	5	1.93	155	0.0167
The Bahamas1	1977	FPTP	1	2.52	38	0.0104
The Bahamas1	1982	FPTP	1	2.03	43	0.0114
The Bahamas1	1987	FPTP	1	2.11	49	0.0096
The Bahamas1	1992	FPTP	1	1.99	49	0.0102
Bangladesh1	1991	FPTP	1	4.61	300	0.0007
Bangladesh1	1996	FPTP	1	3.48	300	0.0009
Barbados1	1966	FPTP	2	2.72	24	0.0306
Barbados2	1971	FPTP	1	1.96	24	0.0212
Barbados2	1976	FPTP	1	2.03	24	0.0205
Barbados2	1981	FPTP	1	2.02	27	0.0183
Barbados2	1986	FPTP	1	1.93	27	0.0191
Barbados2	1991	FPTP	1	2.29	28	0.0155
Barbados2	1994	FPTP	1	2.48	28	0.0144
Barbados2	1999	FPTP	1	1.84	28	0.0194
Belize1	1984	FPTP	1	2.06	28	0.0173
Belize1	1989	FPTP	1	2.07	28	0.0172
Belize1	1993	FPTP	1	2.03	29	0.0169
Belize1	1998	FPTP	1	1.98	29	0.0174
Canada1	1949	FPTP	1	2.83	262	0.0013
Canada1	1953	FPTP	1	2.85	265	0.0013
Canada1	1957	FPTP	1	2.98	265	0.0012
Canada1	1958	FPTP	1	2.44	265	0.0015
Canada1	1962	FPTP	1	3.23	265	0.0011
Canada1	1963	FPTP	1	3.2	265	0.0011
Canada1	1965	FPTP	1	3.31	265	0.0011
Canada1	1968	FPTP	1	2.97	264	0.0012

Table 5.2: Aggregated threshold values for $V^{nec}_{S_T=1}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=1}$
Canada1	1972	FPTP	1	3.25	264	0.00116
Canada1	1974	FPTP	1	2.96	264	0.0012
Canada1	1979	FPTP	1	3.09	282	0.0011
Canada1	1980	FPTP	1	2.93	282	0.0012
Canada1	1984	FPTP	1	2.75	282	0.0012
Canada1	1988	FPTP	1	3.04	295	0.0011
Canada1	1993	FPTP	1	3.92	295	0.0008
Canada1	1997	FPTP	1	4.08	301	0.0008
Canada1	2000	FPTP	1	3.77	301	0.0008
Central African Rep.1	1998	TRS	1	4.19	109	0.0021
Comoros1	1992	TRS	1	14.89	42	0.0015
Dominica1	1980	FPTP	1	2.95	21	0.0161
Dominica1	1985	FPTP	1	2.1	21	0.0226
Dominica1	1990	FPTP	1	2.69	21	0.0177
Dominica1	1995	FPTP	1	2.99	21	0.0159
Dominica1	2000	FPTP	1	2.55	21	0.0186
France2	1958	TRS	1	6.08	465	0.0003
France2	1962	TRS	1	4.93	465	0.0004
France2	1967	TRS	1	4.55	470	0.0004
France2	1968	TRS	1	4.32	470	0.0004
France2	1973	TRS	1	5.68	473	0.0003
France2	1978	TRS	1	5.08	474	0.0004
France2	1981	TRS	1	4.14	474	0.0005
France4	1988	TRS	1	4.38	555	0.0004
France4	1993	TRS	1	6.71	577	0.0002
France4	1997	TRS	1	6.54	577	0.0002
Ghana1	1979	FPTP	1	3.75	140	0.0019
Greece1	1952	Plurality	3	2.7	300	0.0037
Grenada1	1976	FPTP	1	2.07	15	0.0322
Grenada2	1984	FPTP	1	2.11	15	0.0315
Grenada2	1990	FPTP	1	3.85	15	0.0173
Grenada2	1995	FPTP	1	3.65	15	0.0182

Table 5.2: Aggregated threshold values for $V^{nec}_{S_T=1}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=1}$
Grenada2	1999	FPTP	1	2.15	15	0.0310
India1	1952	FPTP	1	3.82	479	0.0005
India1	1957	FPTP	1	3.4	482	0.0006
India1	1962	FPTP	1	4.15	491	0.0004
India1	1967	FPTP	1	4.67	515	0.0004
India1	1971	FPTP	1	4.38	517	0.0004
India1	1977	FPTP	1	3.35	540	0.0005
India1	1980	FPTP	1	4.13	528	0.0004
India1	1984	FPTP	1	3.83	541	0.0004
India1	1989	FPTP	1	4.74	529	0.0003
India1	1991	FPTP	1	5.08	534	0.0003
India1	1996	FPTP	1	5.38	543	0.0003
India1	1999	FPTP	1	6.7	543	0.0002
Jamaica1	1962	FPTP	1	2.06	45	0.0107
Jamaica1	1967	FPTP	1	2.01	53	0.0093
Jamaica1	1972	FPTP	1	1.97	53	0.0095
Jamaica1	1976	FPTP	1	1.96	60	0.0085
Jamaica1	1980	FPTP	1	1.94	60	0.0085
Jamaica1	1983	FPTP	1	1.23	60	0.0135
Jamaica1	1989	FPTP	1	1.97	60	0.0084
Jamaica1	1993	FPTP	1	1.91	60	0.0087
Jamaica1	1997	FPTP	1	2.14	60	0.0077
South Korea1	1960	FPTP	1	2.52	233	0.0017
Macedonia1	1994	TRS	1	3.2	120	0.0026
Malawi1	1994	FPTP	1	2.74	177	0.0020
Malawi1	1999	FPTP	1	2.82	192	0.0018
Mali1	1992	TRS	2	3.56	116	0.0048
Mali2	1997	TRS	3	1.59	147	0.0128
Mongolia1	1992	BV	3	2.71	76	0.0145
Mongolia2	1996	TRS	1	2.58	76	0.0050
Mongolia2	2000	TRS	1	3.24	76	0.0040
Myanmar1	1956	FPTP	1	3.05	250	0.0013

Table 5.2: Aggregated threshold values for $V^{nec}_{S_T=1}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=1}$
Myanmar2	1960	FPTP	3	2.41	250	0.0049
Nepal1	1991	FPTP	1	3.89	205	0.0012
Nepal1	1994	FPTP	1	3.83	205	0.0012
Nepal1	1999	FPTP	1	3.89	205	0.0012
New Zealand1	1946	FPTP	1	2.01	80	0.0062
New Zealand1	1949	FPTP	1	2.03	80	0.0061
New Zealand1	1951	FPTP	1	1.99	80	0.0062
New Zealand1	1954	FPTP	1	2.48	80	0.0050
New Zealand1	1957	FPTP	1	2.3	80	0.0054
New Zealand1	1960	FPTP	1	2.37	80	0.0052
New Zealand1	1963	FPTP	1	2.39	80	0.0052
New Zealand1	1966	FPTP	1	2.61	80	0.0047
New Zealand1	1969	FPTP	1	2.45	84	0.0048
New Zealand1	1972	FPTP	1	2.43	87	0.0047
New Zealand1	1975	FPTP	1	2.55	87	0.0045
New Zealand1	1978	FPTP	1	2.87	92	0.0037
New Zealand1	1981	FPTP	1	2.89	92	0.0037
New Zealand1	1984	FPTP	1	2.99	95	0.0035
New Zealand1	1987	FPTP	1	2.34	97	0.0044
New Zealand1	1990	FPTP	1	2.78	97	0.0037
New Zealand1	1993	FPTP	1	3.52	99	0.0028
Nigeria1	1964	FPTP	1	3.91	469	0.0005
Nigeria2	1979	FPTP	1	3.71	449	0.0006
Nigeria3	1999	FPTP	1	2.32	360	0.0011
Pakistan1	1977	FPTP	1	2.04	200	0.0024
Pakistan1	1988	FPTP	1	3.57	207	0.0013
Pakistan1	1990	FPTP	1	3.44	207	0.0014
Pakistan1	1993	FPTP	1	3.2	207	0.0015
Pakistan1	1997	FPTP	1	3.53	207	0.0013
Philippine1	1946	FPTP	1	3.32	98	0.0030
Philippine1	1949	FPTP	1	2.44	100	0.0040
Philippine1	1953	FPTP	1	2.56	102	0.0038

Table 5.2: Aggregated threshold values for $V^{nec}_{S_T=1}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=1}$
Philippine1	1957	FPTP	1	2.14	102	0.0045
Philippine1	1961	FPTP	1	2.05	104	0.0046
Philippine1	1965	FPTP	1	2.26	104	0.0042
Philippine2	1992	FPTP	1	4.52	200	0.0011
Philippine2	1995	FPTP	1	4.69	204	0.0010
Sierra Leone1	1962	FPTP	1	2.99	62	0.0053
Sierra Leone1	1967	FPTP	1	2.69	66	0.0056
Solomon Islands1	1980	FPTP	1	2.53	38	0.0104
Solomon Islands1	1984	FPTP	1	4.19	38	0.0062
Solomon Islands2	1993	FPTP	1	5.22	47	0.0040
Sri Lanka1	1952	FPTP	1	3.84	89	0.0029
Sri Lanka1	1956	FPTP	1	3.79	89	0.0029
Sri Lanka2	1960	FPTP	1	3.69	145	0.0018
Sri Lanka2	1965	FPTP	1	3.83	145	0.0018
Sri Lanka2	1970	FPTP	1	3.4	145	0.0020
Sri Lanka2	1977	FPTP	1	2.8	168	0.0021
St. Kitts1	1984	FPTP	1	2.45	11	0.0371
St. Kitts1	1989	FPTP	1	2.77	11	0.0328
St. Kitts1	1993	FPTP	1	3.08	11	0.0295
St. Kitts1	1995	FPTP	1	2.65	11	0.0343
St. Kitts1	2000	FPTP	1	2.55	11	0.0356
St. Lucia1	1979	FPTP	1	1.97	17	0.0298
St. Lucia1	1982	FPTP	1	2.4	17	0.0245
St. Lucia1	1987	FPTP	1	2.32	17	0.0253
St. Lucia1	1987	FPTP	1	2.21	17	0.0266
St. Lucia1	1992	FPTP	1	1.97	17	0.0298
St. Lucia1	1997	FPTP	1	1.95	17	0.0301
St. Vincent1	1979	FPTP	1	2.57	13	0.0299
St. Vincent1	1984	FPTP	1	2.28	13	0.0337
St. Vincent1	1989	FPTP	1	1.88	15	0.0354
St. Vincent1	1994	FPTP	1	2.48	15	0.0268
St. Vincent1	1998	FPTP	1	1.99	15	0.0335

Table 5.2: Aggregated threshold values for $V^{nec}_{S_T=1}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=1}$
Thailand1	1983	BV	2	5.72	324	0.0010
Thailand2	1986	BV	3	7.97	347	0.0007
Thailand2	1988	BV	3	9.75	357	0.0008
Thailand2	1992	BV	3	6.7	360	0.0012
Thailand2	1992	BV	3	6.6	360	0.0012
Thailand2	1995	BV	3	6.81	391	0.0011
Thailand2	1996	BV	3	4.61	393	0.0016
Trinidad and Tobago1	1966	FPTP	1	2.51	36	0.011
Trinidad and Tobago1	1971	FPTP	1	1.38	36	0.020
Trinidad and Tobago1	1976	FPTP	1	2.65	36	0.0104
Trinidad and Tobago1	1981	FPTP	1	2.82	36	0.0098
Trinidad and Tobago1	1986	FPTP	1	1.84	36	0.0150
Trinidad and Tobago1	1991	FPTP	1	2.88	36	0.0096
Trinidad and Tobago1	1995	FPTP	1	2.22	36	0.0125
Trinidad and Tobago1	2000	FPTP	1	2.15	36	0.0129
Uganda1	1980	FPTP	1	2.24	126	0.0035
Ukraine1	1994	TRS	1	2.16	450	0.0010
United Kingdom1	1950	FPTP	1	2.44	625	0.0006
United Kingdom1	1951	FPTP	1	2.13	625	0.0007
United Kingdom1	1955	FPTP	1	2.16	630	0.0007
United Kingdom1	1959	FPTP	1	2.28	630	0.0006
United Kingdom1	1964	FPTP	1	2.52	630	0.0006
United Kingdom1	1966	FPTP	1	2.42	630	0.0006
United Kingdom1	1970	FPTP	1	2.46	630	0.0006
United Kingdom1	1974	FPTP	1	3.13	635	0.0005
United Kingdom1	1974	FPTP	1	3.15	635	0.0004
United Kingdom1	1979	FPTP	1	2.87	635	0.0005
United Kingdom1	1983	FPTP	1	2.83	650	0.0005
United Kingdom1	1987	FPTP	1	2.85	650	0.0005
United Kingdom1	1992	FPTP	1	3.03	651	0.0005
United Kingdom1	1997	FPTP	1	3.21	659	0.0004
United States1	1946	FPTP	1	2.05	432	0.0011

Table 5.2: Aggregated threshold values for $V^{nec}_{S_T=1}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=1}$
United States1	1948	FPTP	1	2.07	433	0.0011
United States1	1950	FPTP	1	2.06	434	0.0011
United States1	1952	FPTP	1	2.04	433	0.0011
United States1	1954	FPTP	1	2.01	432	0.0011
United States1	1956	FPTP	1	2.01	435	0.0011
United States1	1958	FPTP	1	1.98	433	0.0011
United States1	1960	FPTP	1	2.01	436	0.0011
United States1	1962	FPTP	1	2.01	437	0.0011
United States1	1964	FPTP	1	1.97	435	0.0011
United States1	1966	FPTP	1	2.03	435	0.0011
United States1	1968	FPTP	1	2.05	434	0.0011
United States1	1970	FPTP	1	2.03	435	0.0011
United States1	1972	FPTP	1	2.04	434	0.0011
United States1	1974	FPTP	1	2	431	0.0011
United States1	1976	FPTP	1	2.02	435	0.0011
United States1	1978	FPTP	1	2.03	435	0.0011
United States1	1980	FPTP	1	2.06	433	0.0011
United States1	1982	FPTP	1	2.02	435	0.0011
United States1	1984	FPTP	1	2.03	434	0.0011
United States1	1986	FPTP	1	2.01	435	0.0011
United States1	1988	FPTP	1	2.03	435	0.0011
United States1	1990	FPTP	1	2.07	435	0.0011
United States1	1992	FPTP	1	2.14	434	0.0010
United States1	1994	FPTP	1	2.08	434	0.0011
United States1	1996	FPTP	1	2.11	434	0.0010
United States1	1998	FPTP	1	2.15	434	0.0010
United States1	2000	FPTP	1	2.18	435	0.0010
Zambia1	1991	FPTP	1	1.63	150	0.0040
Zambia1	1996	FPTP	1	2.44	150	0.0027

The values shown in Table 5.2 clearly reveal the majoritarian character of these electoral systems. This table shows how easy it is for a party with strong support to win a seat in a given district. Of course, there are interesting variations in the values that are derived directly from the function used to calculate them. Electoral systems with a limited number of parties and a small assembly produce higher values than electoral systems with a larger number of parties and larger assemblies. A couple of examples serve to illustrate this point. In the general election held in Antigua in 1994, a parliament of 17 members was elected using the FPTP electoral formula. The number of parties that competed in that election was 2.07. The aggregated threshold value to win 1 seat in that electoral system is 0.0284, meaning that at least 2.84% of the vote was required to win a seat. This value can be contrasted with that resulting from the electoral system used in the 1991 general election in India. Here 534 member were elected also using FPTP and about 5.08 parties competed for those seats. In this case, the aggregated threshold value for winning 1 seat was just 0.0003, in other words, 0.03% of the vote.

More revealing than these results are those for the cases in which $S_T = \frac{M}{2}$. In these cases, one can clearly see the institutionally-determined condition to win the majority in the parliament. The following table shows these results.

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
Antigua1	1984	FPTP	1	1.92	17	0.26041
Antigua1	1989	FPTP	1	1.98	17	0.2525
Antigua1	1994	FPTP	1	2.07	17	0.2415
Antigua1	1999	FPTP	1	2.09	17	0.2392
Argentina1	1951	Plurality	11	1.97	158	0.2538
Argentina2	1954	Plurality	5	1.93	155	0.2590
The Bahamas1	1977	FPTP	1	2.52	38	0.1984
The Bahamas1	1982	FPTP	1	2.03	43	0.2463

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
The Bahamas1	1987	FPTP	1	2.11	49	0.2369
The Bahamas1	1992	FPTP	1	1.99	49	0.2512
Bangladesh1	1991	FPTP	1	4.61	300	0.1084
Bangladesh1	1996	FPTP	1	3.48	300	0.1436
Barbados1	1966	FPTP	2	2.72	24	0.1838
Barbados2	1971	FPTP	1	1.96	24	0.2551
Barbados2	1976	FPTP	1	2.03	24	0.2463
Barbados2	1981	FPTP	1	2.02	27	0.2475
Barbados2	1986	FPTP	1	1.93	27	0.2590
Barbados2	1991	FPTP	1	2.29	28	0.2183
Barbados2	1994	FPTP	1	2.48	28	0.2016
Barbados2	1999	FPTP	1	1.84	28	0.2717
Belize1	1984	FPTP	1	2.06	28	0.2427
Belize1	1989	FPTP	1	2.07	28	0.2415
Belize1	1993	FPTP	1	2.03	29	0.2463
Belize1	1998	FPTP	1	1.98	29	0.2525
Canada1	1949	FPTP	1	2.83	262	0.1766
Canada1	1953	FPTP	1	2.85	265	0.1754
Canada1	1957	FPTP	1	2.98	265	0.1677
Canada1	1958	FPTP	1	2.44	265	0.2049
Canada1	1962	FPTP	1	3.23	265	0.1547
Canada1	1963	FPTP	1	3.2	265	0.1562
Canada1	1965	FPTP	1	3.31	265	0.1510
Canada1	1968	FPTP	1	2.97	264	0.1683
Canada1	1972	FPTP	1	3.25	264	0.1538
Canada1	1974	FPTP	1	2.96	264	0.1689
Canada1	1979	FPTP	1	3.09	282	0.1618
Canada1	1980	FPTP	1	2.93	282	0.1706
Canada1	1984	FPTP	1	2.75	282	0.1818
Canada1	1988	FPTP	1	3.04	295	0.1644
Canada1	1993	FPTP	1	3.92	295	0.1275

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
Canada1	1997	FPTP	1	4.08	301	0.1225
Canada1	2000	FPTP	1	3.77	301	0.1326
Central African Rep.1	1998	TRS	1	4.19	109	0.1193
Comoros1	1992	TRS	1	14.89	42	0.0335
Dominica1	1980	FPTP	1	2.95	21	0.1694
Dominica1	1985	FPTP	1	2.1	21	0.2380
Dominica1	1990	FPTP	1	2.69	21	0.1858
Dominica1	1995	FPTP	1	2.99	21	0.1672
Dominica1	2000	FPTP	1	2.55	21	0.1960
France2	1958	TRS	1	6.08	465	0.08223
France2	1962	TRS	1	4.93	465	0.1014
France2	1967	TRS	1	4.55	470	0.1098
France2	1968	TRS	1	4.32	470	0.1157
France2	1973	TRS	1	5.68	473	0.0880
France2	1978	TRS	1	5.08	474	0.0984
France2	1981	TRS	1	4.14	474	0.1207
France4	1988	TRS	1	4.38	555	0.1141
France4	1993	TRS	1	6.71	577	0.0745
France4	1997	TRS	1	6.54	577	0.0764
Ghana1	1979	FPTP	1	3.75	140	0.1333
Greece1	1952	Plurality	3	2.7	300	0.1851
Grenada1	1976	FPTP	1	2.07	15	0.2415
Grenada2	1984	FPTP	1	2.11	15	0.2369
Grenada2	1990	FPTP	1	3.85	15	0.1298
Grenada2	1995	FPTP	1	3.65	15	0.1369
Grenada2	1999	FPTP	1	2.15	15	0.2325
India1	1952	FPTP	1	3.82	479	0.1308
India1	1957	FPTP	1	3.4	482	0.1470
India1	1962	FPTP	1	4.15	491	0.1204
India1	1967	FPTP	1	4.67	515	0.1070
India1	1971	FPTP	1	4.38	517	0.1141

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
India1	1977	FPTP	1	3.35	540	0.1492
India1	1980	FPTP	1	4.13	528	0.1210
India1	1984	FPTP	1	3.83	541	0.1305
India1	1989	FPTP	1	4.74	529	0.1054
India1	1991	FPTP	1	5.08	534	0.0984
India1	1996	FPTP	1	5.38	543	0.0929
India1	1999	FPTP	1	6.7	543	0.0746
Jamaica1	1962	FPTP	1	2.06	45	0.2427
Jamaica1	1967	FPTP	1	2.01	53	0.2487
Jamaica1	1972	FPTP	1	1.97	53	0.2538
Jamaica1	1976	FPTP	1	1.96	60	0.2551
Jamaica1	1980	FPTP	1	1.94	60	0.2577
Jamaica1	1983	FPTP	1	1.23	60	0.4065
Jamaica1	1989	FPTP	1	1.97	60	0.2538
Jamaica1	1993	FPTP	1	1.91	60	0.2617
Jamaica1	1997	FPTP	1	2.14	60	0.2336
South Korea1	1960	FPTP	1	2.52	233	0.1984
Macedonia1	1994	TRS	1	3.2	120	0.1562
Malawi1	1994	FPTP	1	2.74	177	0.1824
Malawi1	1999	FPTP	1	2.82	192	0.1773
Mali1	1992	TRS	2	3.56	116	0.1404
Mali2	1997	TRS	3	1.59	147	0.3144
Mongolia1	1992	BV	3	2.71	76	0.1845
Mongolia2	1996	TRS	1	2.58	76	0.1937
Mongolia2	2000	TRS	1	3.24	76	0.1543
Myanmar1	1956	FPTP	1	3.05	250	0.1639
Myanmar2	1960	FPTP	3	2.41	250	0.2074
Nepal1	1991	FPTP	1	3.89	205	0.1285
Nepal1	1994	FPTP	1	3.83	205	0.1305
Nepal1	1999	FPTP	1	3.89	205	0.1285
New Zealand1	1946	FPTP	1	2.01	80	0.2487

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
New Zealand1	1949	FPTP	1	2.03	80	0.2463
New Zealand1	1951	FPTP	1	1.99	80	0.2512
New Zealand1	1954	FPTP	1	2.48	80	0.2016
New Zealand1	1957	FPTP	1	2.3	80	0.2173
New Zealand1	1960	FPTP	1	2.37	80	0.2109
New Zealand1	1963	FPTP	1	2.39	80	0.2092
New Zealand1	1966	FPTP	1	2.61	80	0.1915
New Zealand1	1969	FPTP	1	2.45	84	0.2040
New Zealand1	1972	FPTP	1	2.43	87	0.2057
New Zealand1	1975	FPTP	1	2.55	87	0.1960
New Zealand1	1978	FPTP	1	2.87	92	0.1742
New Zealand1	1981	FPTP	1	2.89	92	0.17301
New Zealand1	1984	FPTP	1	2.99	95	0.1672
New Zealand1	1987	FPTP	1	2.34	97	0.2136
New Zealand1	1990	FPTP	1	2.78	97	0.1798
New Zealand1	1993	FPTP	1	3.52	99	0.1420
Nigeria1	1964	FPTP	1	3.91	469	0.1278
Nigeria2	1979	FPTP	1	3.71	449	0.1347
Nigeria3	1999	FPTP	1	2.32	360	0.2155
Pakistan1	1977	FPTP	1	2.04	200	0.2450
Pakistan1	1988	FPTP	1	3.57	207	0.1400
Pakistan1	1990	FPTP	1	3.44	207	0.1453
Pakistan1	1993	FPTP	1	3.2	207	0.1562
Pakistan1	1997	FPTP	1	3.53	207	0.1416
Philippine1	1946	FPTP	1	3.32	98	0.1506
Philippine1	1949	FPTP	1	2.44	100	0.2049
Philippine1	1953	FPTP	1	2.56	102	0.1953
Philippine1	1957	FPTP	1	2.14	102	0.23364
Philippine1	1961	FPTP	1	2.05	104	0.2439
Philippine1	1965	FPTP	1	2.26	104	0.2212
Philippine2	1992	FPTP	1	4.52	200	0.1106

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
Philippine2	1995	FPTP	1	4.69	204	0.1066
Sierra Leone1	1962	FPTP	1	2.99	62	0.1672
Sierra Leone1	1967	FPTP	1	2.69	66	0.1858
Solomon Islands1	1980	FPTP	1	2.53	38	0.19762
Solomon Islands1	1984	FPTP	1	4.19	38	0.1193
Solomon Islands2	1993	FPTP	1	5.22	47	0.0957
Sri Lanka1	1952	FPTP	1	3.84	89	0.1302
Sri Lanka1	1956	FPTP	1	3.79	89	0.1319
Sri Lanka2	1960	FPTP	1	3.69	145	0.1355
Sri Lanka2	1965	FPTP	1	3.83	145	0.1305
Sri Lanka2	1970	FPTP	1	3.4	145	0.1470
Sri Lanka2	1977	FPTP	1	2.8	168	0.1785
St. Kitts1	1984	FPTP	1	2.45	11	0.2040
St. Kitts1	1989	FPTP	1	2.77	11	0.1805
St. Kitts1	1993	FPTP	1	3.08	11	0.1623
St. Kitts1	1995	FPTP	1	2.65	11	0.1886
St. Kitts1	2000	FPTP	1	2.55	11	0.1960
St. Lucia1	1979	FPTP	1	1.97	17	0.2538
St. Lucia1	1982	FPTP	1	2.4	17	0.2083
St. Lucia1	1987	FPTP	1	2.32	17	0.2155
St. Lucia1	1987	FPTP	1	2.21	17	0.2262
St. Lucia1	1992	FPTP	1	1.97	17	0.2538
St. Lucia1	1997	FPTP	1	1.95	17	0.2564
St. Vincent1	1979	FPTP	1	2.57	13	0.1945
St. Vincent1	1984	FPTP	1	2.28	13	0.2192
St. Vincent1	1989	FPTP	1	1.88	15	0.2659
St. Vincent1	1994	FPTP	1	2.48	15	0.2016
St. Vincent1	1998	FPTP	1	1.99	15	0.2512
Thailand1	1983	BV	2	5.72	324	0.08741
Thailand2	1986	BV	3	7.97	347	0.06273
Thailand2	1988	BV	3	9.75	357	0.05128

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
Thailand2	1992	BV	3	6.6	360	0.07575
Thailand2	1995	BV	3	6.81	391	0.07342
Thailand2	1996	BV	3	4.61	393	0.1084
Trinidad and Tobago1	1966	FPTP	1	2.51	36	0.1992
Trinidad and Tobago1	1971	FPTP	1	1.38	36	0.3623
Trinidad and Tobago1	1976	FPTP	1	2.65	36	0.1886
Trinidad and Tobago1	1981	FPTP	1	2.82	36	0.1773
Trinidad and Tobago1	1986	FPTP	1	1.84	36	0.2717
Trinidad and Tobago1	1991	FPTP	1	2.88	36	0.1736
Trinidad and Tobago1	1995	FPTP	1	2.22	36	0.2252
Trinidad and Tobago	2000	FPTP	1	2.15	36	0.2325
Uganda1	1980	FPTP	1	2.24	126	0.2232
Ukraine1	1994	TRS	1	2.16	450	0.2314
United Kingdom1	1950	FPTP	1	2.44	625	0.2049
United Kingdom1	1951	FPTP	1	2.13	625	0.2347
United Kingdom1	1955	FPTP	1	2.16	630	0.2314
United Kingdom1	1959	FPTP	1	2.28	630	0.2192
United Kingdom1	1964	FPTP	1	2.52	630	0.1984
United Kingdom1	1966	FPTP	1	2.42	630	0.2066
United Kingdom1	1970	FPTP	1	2.46	630	0.2032
United Kingdom1	1974	FPTP	1	3.13	635	0.1597
United Kingdom1	1974	FPTP	1	3.15	635	0.1587
United Kingdom1	1979	FPTP	1	2.87	635	0.1742
United Kingdom1	1983	FPTP	1	2.83	650	0.1766
United Kingdom1	1987	FPTP	1	2.85	650	0.1754
United Kingdom1	1992	FPTP	1	3.03	651	0.1650
United Kingdom1	1997	FPTP	1	3.21	659	0.1557
United States1	1946	FPTP	1	2.05	432	0.24390
United States1	1948	FPTP	1	2.07	433	0.2415
United States1	1950	FPTP	1	2.06	434	0.2427
United States1	1952	FPTP	1	2.04	433	0.2450

Table 5.3: Aggregated threshold values for $V^{nec}_{S_T=\frac{M}{2}}$ (cont.)

Country	**Year**	**Formula**	$\mathbf{M}_d$	**P**	**M**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
United States1	1954	FPTP	1	2.01	432	0.2487
United States1	1956	FPTP	1	2.01	435	0.2487
United States1	1958	FPTP	1	1.98	433	0.2525
United States1	1960	FPTP	1	2.01	436	0.2487
United States1	1962	FPTP	1	2.01	437	0.2487
United States1	1964	FPTP	1	1.97	435	0.2538
United States1	1966	FPTP	1	2.03	435	0.2463
United States1	1968	FPTP	1	2.05	434	0.2439
United States1	1970	FPTP	1	2.03	435	0.2463
United States1	1972	FPTP	1	2.04	434	0.2450
United States1	1974	FPTP	1	2	431	0.25
United States1	1976	FPTP	1	2.02	435	0.2475
United States1	1978	FPTP	1	2.03	435	0.2463
United States1	1980	FPTP	1	2.06	433	0.24271
United States1	1982	FPTP	1	2.02	435	0.2475
United States1	1984	FPTP	1	2.03	434	0.2463
United States1	1986	FPTP	1	2.01	435	0.2487
United States1	1988	FPTP	1	2.03	435	0.2463
United States1	1990	FPTP	1	2.07	435	0.2415
United States1	1992	FPTP	1	2.14	434	0.2336
United States1	1994	FPTP	1	2.08	434	0.2403
United States1	1996	FPTP	1	2.11	434	0.2369
United States1	1998	FPTP	1	2.15	434	0.2325
United States1	2000	FPTP	1	2.18	435	0.2293
Zambia1	1991	FPTP	1	1.63	150	0.3067
Zambia1	1996	FPTP	1	2.44	150	0.2049

Since $S_T = \dfrac{M}{2}$ and $M_d = S_{d,}$ then according to function 5.3 the values shown in Table 5.3 are calculate using the function,

$$V^{nec}_{S_T=\frac{M}{2}} = \frac{M_d(M_d - 1 + c)}{2(M_d - 1 + Pc)} \tag{5.5}$$

When $M_d = 1$, function 5.5 can be simplified as,

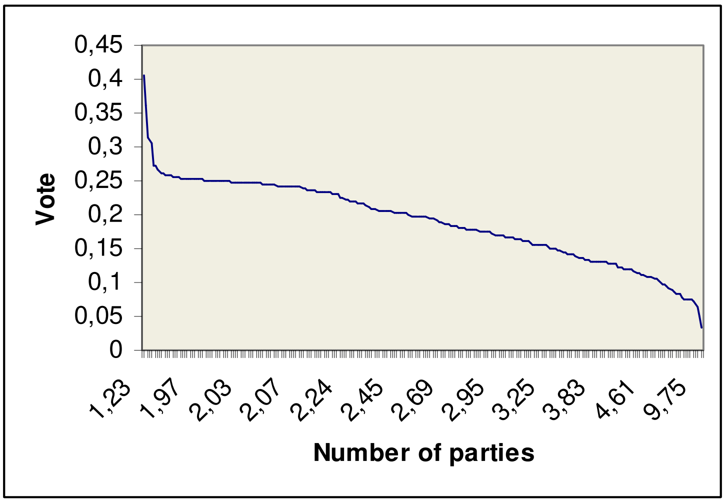
$$V^{nec}_{S_T=\frac{M}{2}} = \frac{1}{2P} \tag{5.6}$$

This new function is very interesting because it shows how the minimum value to win the majority of seats in the parliament depends exclusively on the effective number of parties. So, for example, if the number of parties competing in a complete electoral system with a FPTP electoral formula equals 2, the share of votes below which a party cannot win $S_T = \frac{M}{2}$ seats is 0.25. This situation arises when $S_T = \frac{M}{2}$ seats are won in $\frac{M}{2}$ districts with the minimum number of votes and no votes are won in the remaining districts. This is, of course, an extreme case but it is useful because it gives us the value below which it is impossible to win $S_T = \frac{M}{2}$ seats under any circumstances.

The number of parties, then, is inversely proportional to the number of votes required to win the majority of the seats in the parliament. The greater the number of parties, the smaller the proportion of votes required to win the majority of the seats. Again, this does not necessarily mean that when the number of parties is high it is easier to win a majority in the parliament. What the aggregated values establish is the threshold that must be crossed in order to be in a position to win that number of seats.

Figure 5.1 shows the different aggregated values required to obtain $S_T = \frac{M}{2}$ seats in relation to the number of parties observed in all *winner-takes-all* electoral systems. The graph reveals how the necessary number of votes to win a majority of the seats in the parliament decreases as the number of parties increases. Some of the cases are quite striking. The election held in Jamaica in 1983 produced an effective number of parties of 1.23. The aggregated

Figure 5.1: Aggregated threshold values for $S_T = \frac{M}{2}$ in SMD-electoral systems.



threshold value to win the majority of the seats in the parliament with that number of parties is 40% of the vote. This extremely low number of parties was due to the fact that in this election the majority of the constituencies were not contested, as the opposition boycotted the election because of the country's extreme economic difficulties. In fact, contested elections took place in just six constituencies (Political Handbook of the World 1984-1985). The opposite case is found in Comoros in 1992, when the effective number of parties was 14.89. This extremely high figure was due to the fact that these were the first democratic elections in this African country since it gained independence in 1975 (Keesing's Record of World Events). The aggregated threshold value in this case was 3.35% of the vote.

The data given in this chapter gives us the aggregated threshold valued necessary to win 1 seat in the parliament and a parliamentary majority in the lower chamber. In terms of perfect proportionality, both aggregated values show exactly how disproportional an electoral system with a plurality-majority electoral formula is. If perfect proportionality is understood as implying that a party's share of the vote equals the share of seats it obtains, then the electoral systems analyzed in this chapter are very far from being proportional. As, I have shown, in most cases, 50% of the seats in the parliament were won with a share of 25% or less of the vote. The greater the effective number of parties, the smaller the share of the vote required to win the majority of seats in the parliament.

In the following chapter I will move on to consider electoral systems that use a proportional representation electoral formula. The data analyzed there makes it possible to discover whether these electoral systems are closer to the concept of perfect proportionality as defined above.

Chapter 6

Characterizing Proportional Representation Electoral Systems

In this chapter, I analyze aggregated threshold values for those electoral systems that use a proportional representation electoral formula with party lists. An important distinction between these electoral systems and *winner-takes-all* systems lies in the size of the districts in each type of system. Whereas plurality or majority electoral formulae are applied in electoral systems that normally use single-member districts, proportional representation formulae involve multi-member districts. These methods try to allocate seats in proportion to the share of the vote that each party wins.

There are two different types of party-list proportional representation electoral systems. On the one hand we have proportional representation electoral systems based on a divisor method. Divisor-based proportional representation electoral systems allocate seats in accordance with the results of applying divisors that give the distribution of seats. On the other hand, there are quota-based proportional representation electoral systems. Electoral systems using this type of formula allocate seats in accordance with a fixed predetermined quota of votes.

There are also non-party-list proportional representation electoral systems. Specifically, the Single Transferable Vote (STV) electoral formula uses multimember districts to allocate seats. The difference between this and party-list electoral systems lies in the fact that voters can order their preferences. The ballot for STV is, then, ordinal since it establishes voters´ favorite candidates. As I explained when discussing Single Non Transferable Vote (SNTV), aggregated threshold functions cannot be applied to these type of electoral formula since the process used to allocate seats is carried out in several stages and depends on the actual distribution of votes that each candidate receives. Electoral systems using an STV electoral formula are used most notably in Ireland and Malta.

In the rest of the chapter, I will first offer a broad description of quota-based electoral systems. I will the introduce aggregated threshold data for these electoral systems. A parallel procedure will be used for divisor-based proportional representation electoral systems: after a broad description of the functioning of these electoral systems, I will introduce the aggregated threshold data.

6.1 Quota-based Electoral Systems with Largest Remainders.

The functioning of quota-based electoral systems with largest remainders is very simple. The process used to allocate seats involves two stages. In the first stage, votes are counted and divided by a fixed quota which is obtained using the district magnitude, M_d. The number of full quotas achieved by each political party equals the number of seats that they win. If the number of full quotas obtained by all political parties is smaller than M_d, then, in the second round, the remaining seats are allocated to those parties that get the largest remainders of votes[1].

More formally, a quota, $Q(n)$, is defined as:

$$Q(n) = \frac{V_d}{M_d + n} \tag{6.1}$$

Where M_d is the magnitude of the district d, n is the modifier of the quota and V_d corresponds to the total of valid votes in district d. Considering the total number of parties, P, V_d is logically subjected to,

$$V_d \equiv \sum_{p=1}^{P} V_d^p \text{ for all } p \in \mathbf{P} \tag{6.2}$$

where $\mathbf{P}$ is a $1xP$ vector that contains all competing parties.

The modifier of the quota is an element of a set[2], N, which must satisfy $n > -M_d$. Formally,

$$N = \{n | n \in \mathbf{R} \text{ and } n > -M_d\} \tag{6.3}$$

Dividing the votes for a party p, V_d^p, by the quota, $Q(n)$, yields

$$\frac{V_d^p}{Q(n)} = \frac{V_d^p}{V_d}(M_d + n) \equiv Z_p + r_p \tag{6.4}$$

Where Z_p is an integer between 0 and M and r_p is a fraction ranging from 0 to 1. Mathematically:

$$Z_p \equiv \text{integer}\left(\frac{V_d^p}{V_d}(M_d + n)\right) \tag{6.5}$$

and

[1] For a more detailed account of the working and formulation of quota-based electoral formulae see Penadés 2000:57-65

[2] When $n = 0$ the Hare quota or simple quota is obtained ; when $n = 1$, we obtain the Droop quota and when $n = 2$, the Imperiali quota. The size of the n is important. The largest value of n, i.e., a small quota, the less proportional the electoral formula becomes.

$$r_p = \frac{V_d^p}{V_d}(M_d + n) - Z_p \tag{6.6}$$

If $\sum_{p=1}^{P} Z_p$ for all $p \in \mathbf{P}$ does not equal M_d, then there are R seats to be distributed using the largest remainders method. Hence, a party will get as many seats as full quotas obtain, Z_p, plus one and only one more seat if its remainder, r_p, is one of the R largest remainders. The following example shows how this method works.

Table 6.1: Seat allocation using the Hare quota

$M_d = 5$; $n = 0$ $(Hare)$; $Q(n) = 200$					
Parties	**Votes**	$\frac{V_d^p}{Q(n)}$	Z_p	r_p	**Seats allocated**
A	420	**2.10**	2	0.10	2
B	370	**1.85**	1	0.85	$2(1+1)$
C	180	**0.90**	0	0.90	1
D	30	0.15	0	0.15	0
Total	1000	5	3	2	5

Imagine an electoral system where the total number of votes is split among four parties in a district where 5 seats are to be elected. For simplicity's sake, it is assumed that voting is compulsory and that there is no abstention, so that the sum of the votes won by each party equals the total number of voters. The electoral formula used to allocate seats is one based on the Hare quota. As already explained, this quota is the simplest one since it is obtained by dividing the number of total votes by the size of the district, which in this case is 5. The resulting Hare quota is, then, 200. In the first stage to allocate seats, the votes of each party are divided by the quota. In this example, party A wins 2.10 quotas, party B 1.85, party C 0.90 and party D 0.15 quotas. The number of full quotas amounts to the seats won by each party. So, by looking at those quotients, party A wins 2 seats and party B just 1. Since $M_d = 5$

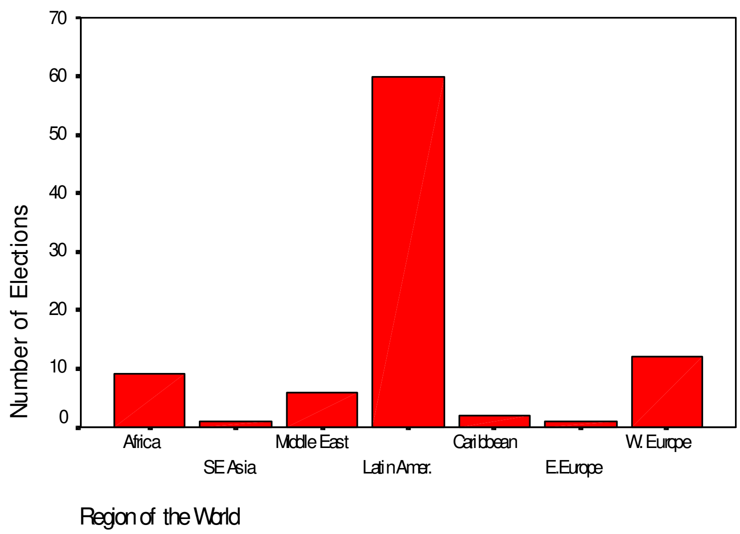
and only 3 seats have already been allocated, there are 2 seats that must be distributed in a second round. In order to do this, the largest remainder method is applied. From Table 6.1 it can be seen that the party with the highest remainder of votes is party C with $r_p = 0.90$ followed by party B, $r_p = 0.85$. These two parties will therefore each receive one of the 2 remaining seats. With this round complete all the seats in the district have been allocated.

As expression 6.1 above shows, the quota depends basically on 2 variables. On the one hand, the size of the district, M_d; on the other hand the modifier of the quota, n. The larger the modifier of the quota or the size of the district, the smaller the quota and vice versa. Smaller quotas produce a bias towards larger parties and larger quotas allow smaller parties to obtain representation. Having explained the workings of quota-based electoral systems, I will now introduce the aggregated threshold data.

6.1.1 Data for Quota-based Electoral Systems.

Quota-based electoral systems are not widely used. Quota-based electoral systems have only been used in 22 democracies in 91 democratic elections between 1945 and 2000. Furthermore, the data suggests that quota-based electoral systems are only used in particular areas of the world. Figure 6.1 shows how these electoral systems are distributed in 7 region of the world.Quota-based electoral systems are most widely used in Latin America. In this region, 11 countries have used this type of electoral system in 60 democratic elections. In Costa Rica this type of electoral formula has been in use continuously since 1953 and in Colombia it has been used since 1974. Quota-based electoral systems have rarely been used in Eastern Europe, in South-East Asia and in the English-speaking Caribbean. In Eastern Europe, only the Slovak Republic has used this method in just one election. In South-East Asia, Indonesia has also used this system once. Among the English-speaking Caribbean countries, Guyana has held two elections using a quota-based electoral system (Golder 2005).

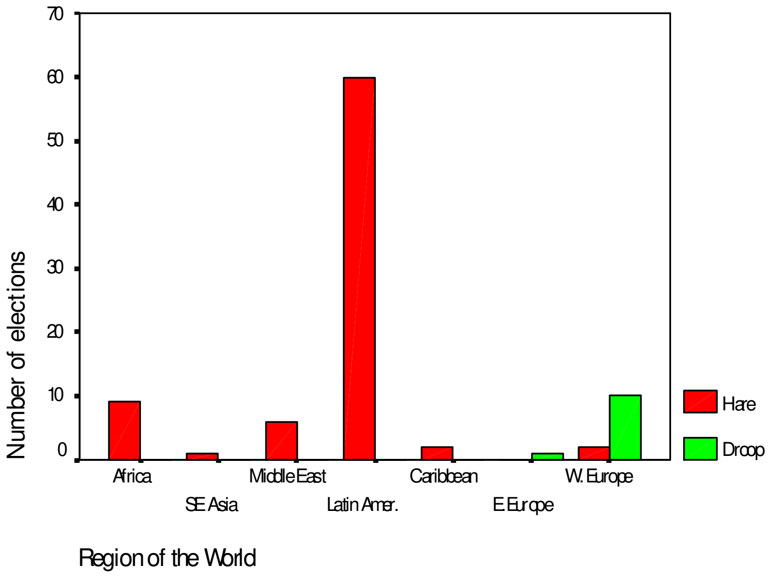
Figure 6.1: World distribution of quota-based electoral systems.



It is also interesting to note that only two types of quota have been used in these democratic elections: Hare and Droop. As Figure 6.2 shows, the Hare quota is by far the most widely used; in contrast only in 11 democratic processes have adopted the Droop quota. This quota was used in the 1998 election in the Slovak Republic and in Luxemburg from 1954 to 1999. If it is curious that quota-based electoral systems are mainly found in Latin America, it is also striking that only the Droop quota is found in Europe. The value of the modifier of the quota for the Droop quota is 1, $n = 1$. This means that those electoral systems that have applied this method to allocate seats produce less proportional results than those applying the Hare quota, $n = 0$, according to expression 6.1 above. Why quota-based electoral systems are more used in Latin America and why a less proportional quota-based electoral systems is used in

Europe is, no doubt, a very interesting question that should be pursued in future research.

Figure 6.2: Quota electoral formulae used between 1945-2000



Data for $S_T = 1$.

The aggregated threshold function for quota-based electoral systems has the following form

$$V_{S_T}^{nec} = \sum_{d=1}^{D} \frac{M_d \left[P(S_d - 1) + 1 + n\right]}{MP(M_d + n)} \tag{6.7}$$

Recall from theorem 2 in Chapter 3 that for any given complete electoral system with a quota-based electoral formula where the number of seats in the parliament is distributed unevenly among all districts, the combination of seats that produces $\min V_{S_T}^{nec}$ for $S_T = 1$ is the following:

a) when $n = 1$, the seat must be won in the smallest district, and

b) when $n = 0$, it does not matter where the seat is won since the value to obtain 1 seat is the same in all districts.

Taking into account theorem 2, expression 6.7 has the following form when $S_T = 1$,

$$V_{S_T=1}^{nec} = \frac{M_d}{M}\left[\frac{1+n}{P(M_d+n)}\right] \tag{6.8}$$

where $M_d = \min M_d \in \mathbf{M}_d$

The results obtained using this function provides the institutionally-determined minimum value below which it is impossible to win 1 seat in the Parliament. It should again be noted that obtaining this value does not guarantee a seat will be won. As I pointed out in Chapter 4, there might be extra institutional settings such as legal thresholds that must be overcome in order to enter the competition for parliamentary seats. When this is the case, the aggregated threshold value and the legal threshold must be contrasted and only the higher value taken into account. In any event, what the aggregated threshold value for $S_T = 1$ indicates is the bottom line that every party must cross if they want to be in the process of seat allocation. Aggregated threshold values for $S_T = 1$ are shown in Table 6.2.

Table 6.2: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for quota-based electoral systems.

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Benin2	1995	18	84	16.34	Hare	0.0007
Benin3	1999	24	83	12.43	Hare	0.0009
Bolivia1	1979	9	117	3.5	Hare	0.0024
Bolivia1	1980	9	130	4.35	Hare	0.0017
Bolivia1	1985	9	130	4.53	Hare	0.0016
Bolivia1	1989	9	130	5.01	Hare	0.0015
Brazil1	1947	22	286	3.59	Hare	0.0009
Brazil6	1998	27	513	8.14	Hare	0.0002
Colombia1	1974	26	200	2.37	Hare	0.0021
Colombia1	1978	26	200	2.17	Hare	0.0023
Colombia1	1982	26	200	2.08	Hare	0.0024
Colombia1	1986	26	200	2.66	Hare	0.0018
Colombia1	1990	26	200	2.2	Hare	0.0022
Colombia2	1991	33	161	3.3	Hare	0.0018
Colombia2	1994	33	161	2.64	Hare	0.0023
Colombia2	1998	33	161	3.5	Hare	0.0017
Costa Rica1	1948	7	45	2.75	Hare	0,0080
Costa Rica1	1953	7	45	2.11	Hare	0.0105
Costa Rica1	1958	7	45	3.57	Hare	0.0062
Costa Rica2	1962	7	57	2.71	Hare	0.0064
Costa Rica2	1966	7	57	2.33	Hare	0.0075
Costa Rica2	1970	7	57	2.56	Hare	0.0068
Costa Rica2	1974	7	57	4.01	Hare	0.0043
Costa Rica2	1978	7	57	2.88	Hare	0.0060
Costa Rica2	1982	7	57	2.53	Hare	0.0069
Costa Rica2	1986	7	57	2.48	Hare	0.0070
Costa Rica2	1990	7	57	2.55	Hare	0.0068
Costa Rica2	1994	7	57	2.77	Hare	0.0063
Costa Rica2	1998	7	57	3.35	Hare	0.0052
El Salvador1	1985	14	60	2.68	Hare	0.0062

Table 6.2: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = 1$ for quota-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=1}$
El Salvador1	1988	14	60	2.75	Hare	0.0060
Guatemala1	1958	22	66	3.23	Hare	0.0046
Guatemala1	1959	22	66	5.29	Hare	0.0028
Guatemala1	1961	22	66	2.89	Hare	0.0052
Guyana1	1992	1	53	2.15	Hare	0.0087
Guyana1	1997	1	53	2.17	Hare	0.0086
Honduras1	1985	18	128	2.14	Hare	0.0036
Honduras1	1989	18	128	2.13	Hare	0.0036
Honduras1	1993	18	128	2.14	Hare	0.0036
Honduras1	1997	18	128	2.43	Hare	0.0032
Israel2	1951	1	120	5.13	Hare	0.0016
Israel2	1955	1	120	6.34	Hare	0.0013
Israel2	1959	1	120	5.17	Hare	0.0016
Israel2	1961	1	120	5.51	Hare	0.0015
Israel2	1965	1	120	4.92	Hare	0.0016
Israel2	1969	1	120	3.63	Hare	0.0022
Luxembourg1	1954	4	52	3	Droop	0,011
Luxembourg1	1959	4	52	3,26	Droop	0,0101
Luxembourg1	1964	4	56	3,5	Droop	0,0087
Luxembourg1	1968	4	56	3,49	Droop	0,0088
Luxembourg1	1974	4	59	4,26	Droop	0,0068
Luxembourg1	1979	4	59	4,16	Droop	0,007
Luxembourg1	1984	4	64	3,56	Droop	0,0077
Luxembourg1	1989	4	60	4.65	Droop	0.0062
Luxembourg1	1994	4	60	4.66	Droop	0.0062
Luxembourg1	1999	4	60	4.62	Droop	0.0063
Namibia1	1994	1	72	1.69	Hare	0.0082
Namibia1	1999	1	72	1.67	Hare	0.0083
Peru3	1980	25	180	4.16	Hare	0.0013
Peru4	1985	26	180	3.02	Hare	0.0018
Peru4	1990	26	180	5.02	Hare	0.0011

Table 6.2: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for quota-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Sierra Leone2	1996	1	68	4.55	Hare	0.0032
Slovak Rep.1	1998	1	150	5.26	Droop	0.0012

The data shown in Table 6.2 corresponds to those cases in which all data is available. As I explained in Chapter 4, this very specific data are required in order to apply aggregated threshold functions. One particularly important element of the system is the data for vector $\mathbf{M}_d$, which indicates the distribution of district magnitudes in the territory. As I showed earlier, when district data is missing a proxy function can be applied. However, whereas this works fairly well for the case of $S_T = \frac{M}{2}$, it does not do so for the case of $S_T = 1$. As a result, no proxy function was used instead in this situation. That said, Table 6.2 includes some 69% of all the quota-based electoral systems. The remaining 31% of missing cases are mainly due to a lack of district data information or a lack of other variables such as the number of parties (1993 election in Madagascar) or even the assembly size and the number of districts (elections in Ecuador between 1952 and 1962)

Also note that when $n = 0$ expression 6.8 is reduced to

$$V_{S_T=1}^{nec} = \frac{1}{MP} \tag{6.9}$$

which is exactly the same function that calculates $V_{S_T=1}^{nec}$ when $M_d = S_d = 1$ in *winner–takes-all* electoral systems as I showed in the previous chapter. Both *winner-takes-all* electoral systems and Hare quota-based electoral systems establish the same aggregated threshold value to win 1 seat in the Parliament. However, note that when the Droop quota is used this result changes. When $n = 1$, expression 6.8 above is reduced to

$$V^{nec}_{S_T=1} = \frac{M_d}{M}\left[\frac{2}{P(M_d+1)}\right] \tag{6.10}$$

and it can be proved that $V^{nec}_{S_T=1}$ when $n = 1$ produces a higher result than $V^{nec}_{S_T=1}$ when $n = 0$. This can be seen using expressions 6.9 and 6.10 above.

$$\frac{M_d}{M}\left[\frac{2}{P(M_d+1)}\right] \geq \frac{1}{MP} \tag{6.11}$$

Simplifying this inequality the following is obtained

$$M_d \geq 1 \tag{6.12}$$

which is always true by definition.

The main conclusion here is not just that Hare quota-based electoral systems produce the same aggregated threshold values to win 1 seat as *winner-takes-all* electoral systems. It is, rather, that Hare quota-based electoral systems produce results that are more favorable to small parties' possibilities of winning a seat because districts are multi-member. Recall from the previous chapter that *winner-takes-all* electoral systems usually have single member districts. Their values always refer to the minimum that large parties in that district must achieve to win a seat. In Hare quota-based electoral systems the aggregated threshold value to win 1 seat refers not only to the large parties but also to medium-size or small ones.

The difference between the values produced by the two quotas used in the database can be seen in Table 6.2. Given similar cases do not exist in reality, a simulation will help to illustrate this idea. Consider the electoral system used in the 1997 election in Guyana. Here, 53 deputies were elected in a single district using the Hare quota. A total of 2.17 effective parties participated in this election. The minimum proportion of the vote required to win 1 seat under this setting was 0.8%. Now suppose that instead of the Hare quota, the electoral formula used to allocate the 53 seats is the Droop quota and everything else remains constant. Under this new setting, the minimum proportion of votes to win 1 seat is 1.6%. This example show how under the Hare quota the probability of small parties entering the parliament is greater than under the Droop quota.

Whereas these values can provide a measure to test the electoral system's potential to incorporate minor parties, the values obtained when applying the aggregated threshold functions when $S_T = \frac{M}{2}$ provide a measure of the difficulties the larger parties must overcome to win a majority of seats in parliament. I present these values in the next section.

Data for $S_T = \frac{M}{2}$

Theorem 5 in Chapter 2 establishes that the combination of seats that produces $\min V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ in a quota-based electoral system where there is an unequal distribution of districts is one in which S_T must be distributed among all districts in $\mathbf{M}_d$. More specifically, seats are distributed first to the smallest districts until these are complete, and then progressively to the larger districts. However, none of these larger districts can be left without representation. Larger districts must have at least 1 seat each.

When the distribution of district is equal such that $M_d = M_D$ for all $M_d \in \mathbf{M}_d$, then, according to theorem 6, the value of $V^{nec}_{S_T=\frac{M}{2}}$ is the same for all possible combinations of seats that produce $S_T = \frac{M}{2}$ if all seats are distributed among all districts. In other words, all combinations of seats that do not include a seat in any of the districts produce a higher value of $V^{nec}_{S_T=\frac{M}{2}}$.

As for $S_T = 1$, the aggregated threshold values for $S_T = \frac{M}{2}$ indicates the value below which the electoral systems does not allow any party to win a majority of the seats in parliament. For any party to win that number of seats it must obtain at least the aggregated threshold value. To put it more simply, these values show the easiest to meet condition to win the majority of the seats in the parliament. With these ideas in mind and the theorems above, the aggregated threshold data is shown in Table 6.3

Table 6.3: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for quota-based electoral systems.

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Benin1	1991	6	64	9,62	Hare		0.4160
Benin2	1995	18	84	16,34	Hare	0.3786	
Benin3	1999	24	83	12,43	Hare	0.3715	
Bolivia1	1979	9	117	3,5	Hare	0.4493	
Bolivia1	1980	9	130	4,35	Hare	0.4467	
Bolivia1	1985	9	130	4,53	Hare	0.4461	
Bolivia1	1989	9	130	5,01	Hare	0.4446	
Brazil1	1947	22	286	3,59	Hare	0.4445	
Brazil6	1998	27	513	8,14	Hare	0.4548	
Colombia1	1974	26	200	2,37	Hare	0.4249	
Colombia1	1978	26	200	2,17	Hare	0.4299	
Colombia1	1982	26	200	2,08	Hare	0.4325	
Colombia1	1986	26	200	2,66	Hare	0.4189	
Colombia1	1990	26	200	2,2	Hare	0.4291	
Colombia2	1991	33	161	3,3	Hare	0.4094	
Colombia2	1994	33	161	2,64	Hare	0.4192	
Colombia2	1998	33	161	3,5	Hare	0.4071	
Costa Rica 1	1948	7	45	2,75	Hare	0.4121	
Costa Rica1	1953	7	45	2,11	Hare	0.4293	
Costa Rica1	1958	7	45	3,57	Hare	0.3991	
Costa Rica2	1962	7	57	2,71	Hare	0.4313	
Costa Rica2	1966	7	57	2,33	Hare	0.4387	
Costa Rica2	1970	7	57	2,56	Hare	0.4339	
Costa Rica2	1974	7	57	4,01	Hare	0.4166	
Costa Rica2	1978	7	57	2,88	Hare	0.4286	
Costa Rica2	1982	7	57	2,53	Hare	0.4345	
Costa Rica2	1986	7	57	2,48	Hare	0.4355	

Table 6.3: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ for quota-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Costa Rica2	1990	7	57	2,55	Hare	0.4341	
Costa Rica2	1994	7	57	2,77	Hare	0.4303	
Costa Rica2	1998	7	57	3,35	Hare	0.4226	
El Salvador1	1985	14	60	2,68	Hare	0.3537	
El Salvador1	1988	14	60	2,75	Hare	0.3515	
Guatemala1	1958	22	66	3,23	Hare	0.2699	
Guatemala1	1959	22	66	5,29	Hare	0.2297	
Guatemala1	1961	22	66	2,89	Hare	0.282	
Guyana1	1992	1	53	2,15	Hare	0.4993	
Guyana 1	1997	1	53	2,17	Hare	0.4993	
Honduras1	1985	18	128	2,14	Hare	0.4251	
Honduras1	1989	18	128	2,13	Hare	0.4254	
Honduras1	1993	18	128	2,14	Hare	0.4251	
Honduras1	1997	18	128	2,43	Hare	0.4185	
Indonesia1	1999	27	462	5,05	Hare		0.4531
Israel2	1951	1	120	5,13	Hare	0.4933	
Israel2	1955	1	120	6,34	Hare	0.493	
Israel2	1959	1	120	5,17	Hare	0.4933	
Israel2	1961	1	120	5,51	Hare	0.4932	
Israel2	1965	1	120	4,92	Hare	0.4934	
Israel2	1969	1	120	3,63	Hare	0.494	
Liechtenstein1	1993	2	25	2,33	Hare		0.4543
Liechtenstein1	1997	2	25	2,33	Hare		0.4543
Luxembourg1	1954	4	52	3	Droop	0.4315	
Luxembourg1	1959	4	52	3,26	Droop	0.4277	
Luxembourg1	1964	4	56	3,5	Droop	0.4286	
Luxembourg1	1968	4	56	3,49	Droop	0.4287	
Luxembourg1	1974	4	59	4,26	Droop	0.4176	
Luxembourg1	1979	4	59	4,16	Droop	0.4183	

Table 6.3: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ for quota-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=\frac{M}{2}}^{nec}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Luxembourg1	1984	4	64	3,56	Droop	0.4366	
Luxembourg1	1989	4	60	4,65	Droop	0.4248	
Luxembourg1	1994	4	60	4,66	Droop	0.4248	
Luxembourg1	1999	4	60	4,62	Droop	0.425	
Namibia1	1994	1	72	1,69	Hare	0.4943	
Namibia1	1999	1	72	1,67	Hare	0.4944	
Nicaragua1	1990	9	90	2,18	Hare		0.4458
Panama1	1960	10	53	4,71	Hare		0.3513
Peru1	1962	24	182	3,39	Hare		0.4069
Peru2	1963	24	139	2,98	Hare		0.3852
Peru3	1980	25	180	4,16	Hare	0.3945	
Peru4	1985	26	180	3,02	Hare	0.4034	
Peru4	1990	26	180	5,02	Hare	0.3843	
Sierra Leone2	1996	1	68	4,55	Hare	0.4885	
Slovak Rep.1	1998	1	150	5,26	Droop	0.4926	
Somalia1	1964	47	123	3,02	Hare		0.2447

Table 6.3 shows aggregated threshold values calculated according to function 6.7 above. It is difficult to say exactly how each variable affects the result of the function. However, Table 6.3 does suggest some intuitions. The lowest aggregated threshold value is 0.2297 and comes from the 1959 general election in Guatemala. The complete electoral system used there had 22 districts, which sent 66 deputies to the lower chamber. According to the definition given in Chapter 4, the average district magnitude given these two variables is 3. The number of parties competing in this election was 5.29. The highest aggregated threshold value is 0.4993 and comes from the 1997 general election in Guyana. During this electoral process, 53 deputies were elected in a single district and an effective number of parties of 2.17 competed for those seats. The average district

magnitude in this case was, then, 53. Looking at the values in the table it seems that higher values are obtained when the number of districts is particularly low.

As Appendix A shows this intuition proves to be true. Theorems 7 and 8 show how the number of districts is inversely proportional to the proportion of the vote required to win the majority of seats in parliament. In other words, the higher the number of districts the lower the minimum proportion of votes required to win $S_T = \frac{M}{2}$ seats in the parliament. These two theorems explain why higher values can be found in those complete electoral systems where the number of districts is close to 1.

Table 6.3 also shows values calculated using a proxy function. As explained in Chapter 4, this function is applied in the event of missing data. This proxy function is applied more specifically when the vector containing all district magnitudes is missing, $\mathbf{M}_d$. For quota-based electoral systems, the proxy function has the following form:

$$\widehat{V_{S_T}^{nec}} = \frac{P(\widehat{S}-1)+1+n}{P(\widehat{M}+n)} \tag{6.13}$$

where $\widehat{M}$ refers to average district magnitude and $\widehat{S}$ refers to average seats. Both concepts are defined in Chapter 4 and are calculated using the following forms,

$$\widehat{M} = \frac{M}{D} \tag{6.14}$$

and

$$\widehat{S} = \frac{\widehat{M}S_T}{M} \tag{6.15}$$

When $S_T = \dfrac{M}{2}$ and $\mathbf{M}_d$ is missing, expression 6.13 above can be simplified as

$$\widehat{V^{nec}_{S_T}} = \frac{P(M-2D)+2D(1+n)}{2P(M+Dn)} \tag{6.16}$$

This proxy function is used for about 12.5% of the cases shown in Table 6.3. As for these cases, it seems that there is a clear pattern relating lower aggregated threshold values to lower average magnitudes.

Quota-based electoral systems account for only about 35% of the cases classified as party-list proportional representation electoral system. The remaining 65% corresponds to divisor-based systems. The next section focuses on these electoral systems.

6.2 Divisor-based Electoral Systems.

A second type of proportional representation electoral formulae is a method based on a criterion of divisors (Penadés 2000: 92-119).The main difference with quota-based electoral formulae is that there is no pre-established size of a divisor (like $M_d + n$ in quota-based methods). A divisor, X, must be found to enable us to calculate the averages needed in order to be allocate the M_d seats[3]. Divisor-based electoral formulae are defined, as Penadés remarks (2000), around the concept of a constant non-negative divisor criterion. This is an adjustment rule

$$c(S^p_d) = S^p_d + c \text{ for } S^p_d > 0 \tag{6.17}$$

where c is the adjustment term. This definition includes any electoral formulae in which the rule to allocate S^p_d seats for any political party p and any divisors X must fulfil,[4]

[3] For a discussion about how to calculate the divisor X see Balinsky and Young (1982) and Penadés (2000)

[4] These rules are pre-established, so for example, for d´Hondt, the criterión is $c(S^p_d) = S^p_d{+}1$. This means that in order to get S^p_d seats, party p must fulfil the following restriction

$S^p_d \leq \frac{V^p_d}{X} \leq S^p_d + 1$

$$S_d^p - 1 + c \leq \frac{V_d^p}{X} \leq S_d^p + c \qquad (6.18)$$

Where c, the adjustment term, is a real number $c \geq 0$ and $\sum_{p=1}^{P} S_d^p = M_d$.

What inequality 6.18 shows is how to adjust seats when the votes of party p divided by divisor X does not produce an integer number. To see the functioning of this method, the following example illustrates how seats are distributed using a divisor-based electoral formula.

Table 6.4: Seat allocation using the Sainte-Laguë algorithm

Parties	**Votes** (V_d^p)	$\mathbf{V_d^p/0.5}$	$\mathbf{V_d^p/1.5}$	$\mathbf{V_d^p/2.5}$	$\mathbf{V_d^p/3.5}$	**Seats**
A	420	**840**	**280**	168	120	2
B	370	**740**	246.6	148	105.7	1
C	180	**360**	120	72	51.4	1
D	30	60	20	12	8.6	0
Total	1000					4

Table 6.4 shows an electoral systems in which 4 political parties compete for 4 seats that are allocated using the Sainte-Laguë electoral formula. The algorithm used to calculate the number of seats each party wins divides the parties total number of votes by a numerical series. In the case of Sainte-Laguë (S-L) this numerical series is: 1, 3, 5, 7 or 0.5, 1.5, 2.5, 3,5. The votes are divided as many times as there are seats to be allocated. Those parties that get the higher averages win that many seats. The table shows how party A has the two highest averages, so wins 2 seats. The remaining two seats are distributed between party B and party C since these two parties also get 1 highest average each.

For Sainte-Laguë is $c(S_d^p) = S_d^p + 0.5$ or
$S_d^p - 0.5 \leq \frac{V_d^p}{X} \leq S_d^p + 0.5$ (See Penadés 2000:86-92)

In order to apply expression 6.18 a value for X must be found. In the case of Sainte-Laguë this value can be the lowest of the winning higher averages as shown in Table 6.4 (Penadés 2000:88). In this case, $X = 280$. Using the X value, the following averages are obtained.

Table 6.5: Seat allocation using a Sainte-Laguë divisor

$M_{d_i} = 4$; $c = 0.5$ (Sainte-Laguë); $X = 280$			
Parties	**Votes**	$\frac{V_d^p}{X}$	**Seats allocated**
A	420	**1.5**	2
B	370	**1.32**	1
C	180	**0.64**	1
D	30	0.10	0
Total	1000		4

Looking at Table 6.5 the following questions can be formulated. Why does party B obtain 1 seat and not 2 like A? Why does party D win 0 instead of 1? These questions are answered precisely by the adjustment rule $c(S_d)$ for $S_d = 1$. To find out the number of seats that party B has won, one just needs to apply this adjustment rule 6.18:

$$1 - 1 + 0.5 \leq 1.3 \leq 1 + 0.5 \tag{6.19}$$

which is equals to

$$0.5 \leq 1.3 \leq 1.5 \tag{6.20}$$

In other words, any $\frac{V_d^p}{X}$ which falls within the interval $[0.5, 1.5]$ will be adjusted as 1. Following the same logic, party A would get 2 seats and party C the remaining 1, leaving party D with no seats.

It is easy to see how the smaller the value of the adjustment term, c, the easier it is to win a seat. Lower values of the adjustment term take the lower bound of the interval closer to 0 and the upper bound of the interval closer to S_d. So, the smaller the value of c, the more proportional the results obtained.

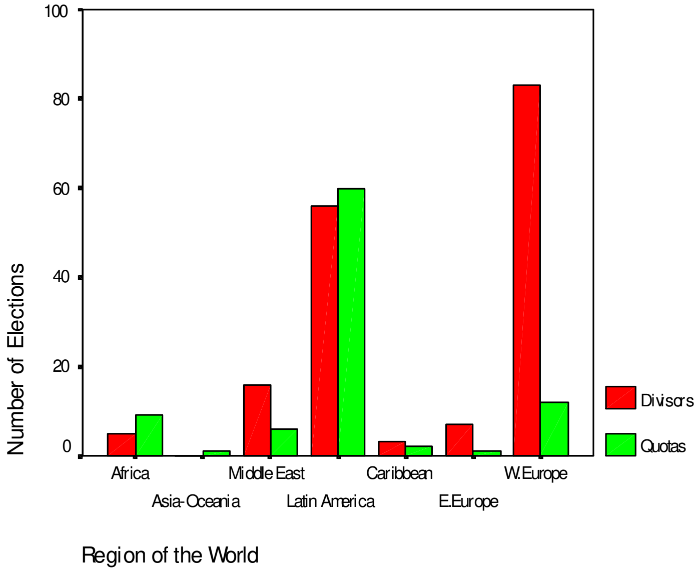
6.2.1 Data for Divisor-based Electoral Systems.

Divisor-based electoral formulae are mostly used in party-list proportional representation electoral systems. In fact, 28 countries used this type of electoral formula in 170 general elections between 1945-2000. Figure 6.3 shows the distribution of both quota- and divisor-based electoral systems in 7 regions of the world. Contrasting with Figure 6.1 above, this chart shows how divisor-based electoral systems are clearly concentrated in Western Europe. Divisor-based electoral systems are used to a similar extent as quota-based systems in Latin America, while the rest of the world rarely uses them. In Western Europe, 9 out of 15 countries included in the database used some type of divisor-based electoral formula between 1945-2000.

Furthermore, Figure 6.4 reveals that the d'Hondt electoral formulae is the most widely used electoral formulae in all the regions of the world where divisors-based electoral systems are adopted. Others divisor-based electoral formulae such as Sainte-Laguë (S-L) and Modified Sainte-Laguë (Mod. S-L) are used only in a few cases. The Sainte-Laguë electoral formula was used, for example, for the general election in Bolivia in 1993, as well as for the 1993, 1995 and 1998 general elections in Latvia. The modified Sainte-Laguë electoral formula is used mainly in the Scandinavian countries. Thus, this method was used in Norway and Sweden between 1953-1985 and 1948-1968 respectively.

The information provided in Figure 6.4 is rich and raises interesting questions for further research. As in quota-based electoral formula, different types of divisor-based formulae produce different degrees of proportionality. So, the d'Hondt electoral formula, $c = 1$, is less proportional than the Sainte-Laguë, $c = 0.5$. Why, then, is d'Hondt the most preferred divisor-based electoral formulae? Why are divisor-based electoral systems mostly adopted in Europe and

Figure 6.3: Quota and divisor-based electoral systems used between 1945-2000



Latin America and not in Asia or Africa? These, again, are important and fundamental questions that deserve further attention in the future.

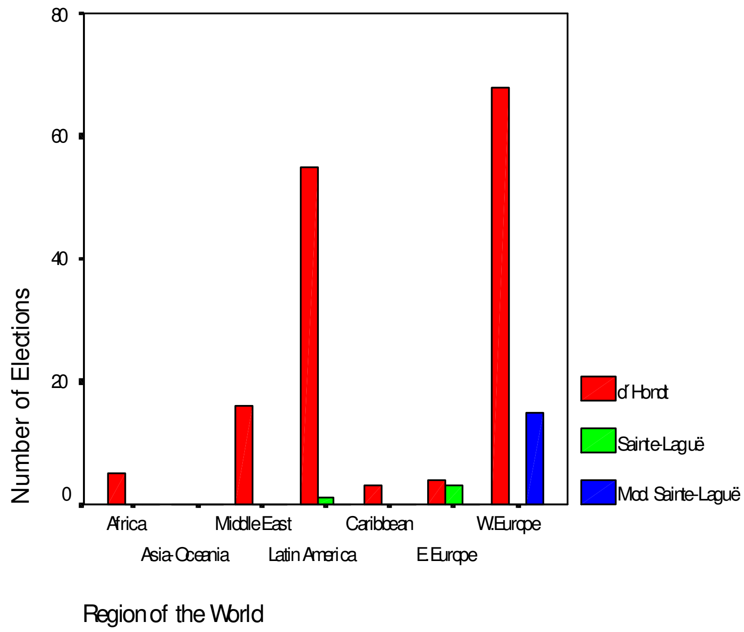
Data for $S_T = 1$

As already mentioned, the aggregated threshold function for divisor-based electoral systems has the following form,

$$V_{S_T}^{nec} = \sum_{d=1}^{D} \frac{M_d}{M} \left(\frac{S_d - 1 + c}{M_d - 1 + Pc} \right) \tag{6.21}$$

Also recall theorem 1 from Chapter 3. This established the combination of seats among all districts that produces $\min V_{S_T}^{nec}$ for

Figure 6.4: Distribution of divisor-based electoral formula around the World between 1945-2000.



$S_T = 1$. More specifically, that theorem establishes that given a complete electoral system with a divisor-based electoral formula, c, and where the number of seats in the parliament, M, is distributed unevenly among all districts, D, the combination of seats that produces $\min V_{S_T}^{nec}$ for $S_T = 1$ is that in which the seat is won in the smallest district.

Considering theorem 1, note that expression 6.21 above adopts the following form when $S_T = 1$,

$$V_{S_T}^{nec} = \frac{M_d}{M} \left(\frac{c}{M_d - 1 + Pc} \right) \tag{6.22}$$

where $M_d = \min M_d \in \mathbf{M}_d$.

The value of this new function gives us the lowest value that any party must obtain in order to be in a position to win a seat in the parliament. Table 6.6 shows these values for those electoral systems that have used divisor-based electoral formula at some time between 1945-2000.

Table 6.6: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = 1$ for divisor-based electoral systems.

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=1}$
Argentina5	1983	24	254	2.63	d´Hondt	0.002
Argentina5	1985	24	127	4.24	d´Hondt	0.003
Argentina6	1987	23	127	3.14	d´Hondt	0.003
Argentina7	1989	24	119	3.35	d´Hondt	0.003
Argentina7	1991	24	130	3.57	d´Hondt	0.003
Argentina7	1993	24	128	3.37	d´Hondt	0.003
Argentina7	1995	24	128	3.58	d´Hondt	0.003
Argentina7	1997	24	127	3.65	d´Hondt	0.003
Argentina7	1999	24	131	3.15	d´Hondt	0.003
Bolivia2	1993	9	130	4.66	S-L	0.0032
Brazil2	1950	25	304	7.11	d´Hondt	0.0004
Brazil2	1954	25	326	8.69	d´Hondt	0.0003
Brazil2	1958	25	326	9.82	d´Hondt	0.0003
Brazil3	1962	25	404	11.17	d´Hondt	0.0002
Brazil4	1982	25	479	2.65	d´Hondt	0.0014
Brazil5	1986	27	495	3.55	d´Hondt	0.0012
Brazil5	1990	27	503	9.79	d´Hondt	0.0009
Brazil5	1994	27	513	8.53	d´Hondt	0.0010
Bulgaria1	1991	31	240	4.12	d´Hondt	0.0023
Bulgaria1	1994	31	240	3.81	d´Hondt	0.0024
Bulgaria1	1997	31	240	3.02	d´Hondt	0.0027
Chile1	1961	28	147	6.44	d´Hondt	0.0010
Chile3	1993	60	120	6.55	d´Hondt	0.0022
Chile3	1997	60	120	6.95	d´Hondt	0.0020

Table 6.6: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Dominican Rep.1	1966	27	74	2.2	d´Hondt	0.0084
Dominican Rep.1	1970	27	74	2.9	d´Hondt	0.0069
Dominican Rep.2	1974	27	91	1.35	d´Hondt	0.0093
Dominican Rep.2	1978	27	91	2.23	d´Hondt	0.0068
Dominican Rep.3	1982	27	120	2.76	d´Hondt	0.0044
Dominican Rep.4	1986	30	120	3.19	d´Hondt	0.0039
Dominican Rep.4	1990	30	120	3.22	d´Hondt	0.0039
Dominican Rep.4	1994	30	120	2.71	d´Hondt	0.0044
Dominican Rep.5	1998	30	150	2.73	d´Hondt	0.0035
Finland1	1948	15	200	4.9	d´Hondt	0.0010
Finland1	1951	15	200	4.96	d´Hondt	0.0010
Finland2	1954	16	200	4.98	d´Hondt	0.0010
Finland2	1958	16	200	5.19	d´Hondt	0.0009
Finland3	1962	15	200	5.86	d´Hondt	0.0008
Finland3	1966	15	200	5.22	d´Hondt	0.0009
Finland3	1970	15	200	6.17	d´Hondt	0.0008
Finland3	1972	15	200	5.95	d´Hondt	0.0008
Finland3	1975	15	200	5.89	d´Hondt	0.0008
Finland3	1979	15	200	5.74	d´Hondt	0.0008
Finland3	1983	15	200	5.45	d´Hondt	0.0009
Finland3	1987	15	200	6.15	d´Hondt	0.0008
Finland3	1991	15	200	5.89	d´Hondt	0.0008
Finland3	1995	15	200	5.77	d´Hondt	0.0008
Finland3	1999	15	200	5.93	d´Hondt	0.0008
Israel1	1949	1	120	5.36	d´Hondt	0.0015
Israel3	1973	1	120	3.81	d´Hondt	0.0021
Israel3	1977	1	120	5.01	d´Hondt	0.0016
Israel3	1981	1	120	3.56	d´Hondt	0.0023
Israel3	1984	1	120	4.3	d´Hondt	0.0019
Israel3	1988	1	120	5.01	d´Hondt	0.0016

Table 6.6: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Israel3	1992	1	120	4.87	d´Hondt	0.0017
Israel3	1996	1	120	5.85	d´Hondt	0.0014
Israel3	1999	1	120	9.74	d´Hondt	0.0008
Latvia1	1993	5	100	6.21	S-L	0.0045
Latvia1	1995	5	100	9.61	S-L	0.0042
Latvia1	1998	5	100	6.94	S-L	0.0044
Moldova1	1998	1	104	5.73	d´Hondt	0.0016
Netherlands1	1946	1	100	4.68	d´Hondt	0.0021
Netherlands1	1948	1	100	4.98	d´Hondt	0.0020
Netherlands1	1952	1	100	4.99	d´Hondt	0.0020
Netherlands2	1956	1	150	4.26	d´Hondt	0.0015
Netherlands2	1959	1	150	4.46	d´Hondt	0.0014
Netherlands2	1963	1	150	4.79	d´Hondt	0.0013
Netherlands2	1967	1	150	6.2	d´Hondt	0.0010
Netherlands2	1971	1	150	7.09	d´Hondt	0.0009
Netherlands2	1972	1	150	6.85	d´Hondt	0.0009
Netherlands2	1977	1	150	3.96	d´Hondt	0.0016
Netherlands2	1981	1	150	4.56	d´Hondt	0.0014
Netherlands2	1982	1	150	4.23	d´Hondt	0.0015
Netherlands2	1986	1	150	3.77	d´Hondt	0.0017
Netherlands2	1989	1	150	3.9	d´Hondt	0.0017
Netherlands2	1994	1	150	5.7	d´Hondt	0.0011
Netherlands2	1998	1	150	5.13	d´Hondt	0.0012
Norway1	1949	29	150	3.76	d´Hondt	0.0034
Norway2	1953	20	150	3.53	Mod.S-L	0.0027
Norway2	1957	20	150	3.44	Mod.S-L	0.0028
Norway2	1961	20	150	3.59	Mod.S-L	0.0027
Norway2	1965	20	150	3.9	Mod.S-L	0.0026
Norway2	1969	20	150	3.61	Mod.S-L	0.0027
Norway3	1973	19	155	4.63	Mod.S-L	0.0024

Table 6.6: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Norway3	1977	19	155	3.85	Mod.S-L	0.0026
Norway3	1981	19	155	3.92	Mod.S-L	0.0026
Norway3	1985	19	155	3.63	Mod.S-L	0.0026
Portugal2	1979	20	246	3	d´Hondt	0.0027
Portugal2	1980	20	246	2.88	d´Hondt	0.0027
Portugal2	1983	20	246	3.73	d´Hondt	0.0024
Portugal2	1985	20	246	4.77	d´Hondt	0.0018
Portugal2	1987	20	246	2.98	d´Hondt	0.0024
Portugal2	1991	20	226	2.86	d´Hondt	0.0027
Portugal2	1995	20	226	3.09	d´Hondt	0.0026
Portugal2	1999	20	226	3.19	d´Hondt	0.0025
Spain1	1977	52	350	4.29	d´Hondt	0.0006
Spain1	1979	52	350	4.25	d´Hondt	0.0006
Spain1	1982	52	350	3.18	d´Hondt	0.0008
Spain1	1986	52	350	3.59	d´Hondt	0.0008
Spain1	1989	52	350	4.08	d´Hondt	0.0007
Spain1	1993	52	350	3.5	d´Hondt	0.0008
Spain1	1996	52	350	3.27	d´Hondt	0.0008
Spain1	2000	52	350	2.99	d´Hondt	0.0009
Suriname1	1991	10	51	2.69	d´Hondt	0.0106
Suriname1	1996	10	51	3.62	d´Hondt	0.0084
Suriname1	2000	10	51	3.77	d´Hondt	0.0082
Sweden1	1948	28	230	3.35	d´Hondt	0.0024
Sweden2	1952	28	230	3.28	Mod.S-L	0.0017
Sweden2	1956	28	231	3.37	Mod.S-L	0.0017
Sweden2	1958	28	231	3.31	Mod.S-L	0.0017
Sweden2	1960	28	232	3.26	Mod.S-L	0.0017
Sweden2	1964	28	233	3.42	Mod.S-L	0.0017
Sweden2	1968	28	233	3.18	Mod.S-L	0.0017
Switzerland1	1947	25	194	5.34	d´Hondt	0.0009

Table 6.6: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = 1$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=1}$
Switzerland1	1951	25	196	5.09	d´Hondt	0.0010
Switzerland1	1957	25	196	4.96	d´Hondt	0.0010
Switzerland1	1959	25	196	5.04	d´Hondt	0.0010
Switzerland1	1963	25	200	5	d´Hondt	0.0010
Switzerland1	1967	25	200	5.54	d´Hondt	0.0009
Switzerland1	1971	25	200	6.16	d´Hondt	0.0008
Switzerland1	1975	25	200	5.78	d´Hondt	0.0008
Switzerland2	1979	26	200	5.5	d´Hondt	0.0009
Switzerland2	1983	26	200	5.99	d´Hondt	0.0008
Switzerland2	1987	26	200	6.8	d´Hondt	0.0007
Switzerland2	1991	26	200	7.34	d´Hondt	0.0006
Switzerland2	1995	26	200	6.82	d´Hondt	0.0007
Switzerland2	1999	26	200	5.86	d´Hondt	0.0008
Turkey4	1995	83	550	6.14	d´Hondt	0.0005
Uruguay1	1946	1	99	3.08	d´Hondt	0.0032
Uruguay1	1950	1	99	2.65	d´Hondt	0.0038
Uruguay1	1954	1	99	2.6	d´Hondt	0.0038
Uruguay1	1958	1	99	2.55	d´Hondt	0.0039
Uruguay1	1962	1	99	2.4	d´Hondt	0.0042
Uruguay1	1966	1	99	2.44	d´Hondt	0.0041
Uruguay1	1971	1	99	2.74	d´Hondt	0.0036
Uruguay2	1989	1	99	3.38	d´Hondt	0.0029
Uruguay2	1994	1	99	3.35	d´Hondt	0.0030
Uruguay2	1999	1	99	3.12	d´Hondt	0.0032

Table 6.6 shows results for 138 general elections held in 17 countries in the period 1945-2000. These values are obtained from those cases in which there exists data for all the variables. Cases with missing data are not included and no proxy function is applied in these cases (for the reasons given above). The values shown in Table 6.6 can be considered as the frontier that must always be crossed to

win 1 seat in the parliament. As for quota-based electoral systems, these values must be considered in contrast with other institutional settings such as legal thresholds. Wherever legal thresholds exist, they must be contrasted with aggregated threshold values. The higher value must be considered as the necessary condition to fulfil in order to win 1 seat.

As is the case in *winner-takes-all* and quota-based electoral systems, the necessary condition to win 1 seat in the parliament is extremely low. The data in Table 6.6 suggests that lower values are obtained when we find a combination of a large parliament, a large number of parties and very small districts. This is the case of the lowest aggregated threshold value obtained from the complete electoral system used in the 1964 general election in Brazil. There, 404 member were elected in 20 districts, the smallest of which was uninominal, and where 11.17 parties competed in an electoral system that used the d'Hondt electoral formula. The aggregated threshold value to win 1 seat produced by all these variables is 0.02%. In contrast, the highest aggregated threshold value is obtained in complete electoral systems with relatively small parliaments and a small number of parties. For example, in Surinam in 1991 51 members of parliament where elected in 10 multimember districts, the smallest of which had size 2. In that election the number of competing parties was 2.69 in a electoral system that also used the d'Hondt electoral formula. The aggregated threshold value to win 1 seat in this complete electoral system was 1.06%. This is a much higher value than 0.02% obtained for the 1964 general election in Brazil.

The effect of the electoral formula cannot be appreciated in Table 6.6 unless a simulation is performed. Take for example the case of the 1993 general election in Latvia. Given that in a complete electoral system where the Sainte-Laguë is used the threshold value for winning 1 seat is 0.45%. Suppose now that, ceteris paribus, the Sainte-Laguë is substituted by the d'Hondt electoral formula. In this case, and holding everything equal, the aggregated threshold value would be about 0.8%, which is significantly higher than the value produced when applying the Sainte-Laguë electoral formula. The small parties are much less likely to win a seat in the parliament in complete electoral systems that use the Sainte-Laguë as opposed to the d'Hondt electoral formula.

Data for $S_T = \dfrac{M}{2}$

The aggregated threshold value when the total number of seats is the majority of the parliament, $S_T = \dfrac{M}{2}$, tells the share of votes below which it is impossible for any party to win that number of seats in the whole territory. This value is obtained from a particular combination of seats. For divisor-based electoral systems, this combination of seats is introduced by theorems 3 and 4 in Chapter 3.

Theorem 3 explains that for a given complete electoral system with a divisor-based electoral formula where the number of seats in the parliament is distributed unequally among all districts, the combination of seats that produces $\min V_{S_T}^{nec}$ when $S_T = \dfrac{M}{2}$ depends on the value of c :

a) If $c = 1$, then the combination of seats that produces $\min V_{S_T}^{nec}$,is one in which the S_T seats are distributed in all small districts. More specifically, seats will be distributed first to the smallest district, then to the second smallest district, and so on until the total number of S_T is reached.

b) For $c = 0.5$ the combination of seats that produces $\min V_{S_T}^{nec}$, is one in which S_T must be distributed among all districts in $\mathbf{M}_{d.}$ Seats are distributed as in a), to the smallest districts until completed and then to the larger districts in succession. However, none of these larger districts can go without representation. Larger districts must have at least 1 seat each.

For those cases in which seats are distributed equally among all districts, i.e. $M_d = M_D$ for all $M_d \in \mathbf{M}_d$, theorem 4 tells us that the combination of seats that produces $\min V_{S_T}^{nec}$ when $S_T = \dfrac{M}{2}$ is obtained when the following conditions hold.

a) The value of $V_{S_T}^{nec}$ is the same for all possible combinations of seats that produce S_T when $c = 1$.

b) However, if $c = 0.5$, then, the value of $V_{S_T}^{nec}$ is the same for all possible combinations of seats that produce S_T if all seats are distributed among all districts. In other words, all particular combinations of seats that do not include a seat in any of the districts produce a higher value of $V_{S_T}^{nec}$.

Applying these two theorems to expression 6.21 above the aggregated threshold values for $S_T = \dfrac{M}{2}$ is obtained. These values are shown in Table 6.7.

Table 6.7: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ for divisor-based electoral systems.

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}_{S_T=\frac{M}{2}}}$
Argentina3	1963	23	192	5.57	d´Hondt		0.3231
Argentina3	1965	23	192	4.87	d´Hondt		0.2593
Argentina4	1973	24	243	3.19	d´Hondt		0.4111
Argentina5	1983	24	254	2.63	d´Hondt	0.3993	
Argentina5	1985	24	127	4.24	d´Hondt	0.2549	
Argentina6	1987	23	127	3.14	d´Hondt	0.3114	
Argentina7	1989	24	119	3.35	d´Hondt	0.2909	
Argentina7	1991	24	130	3.57	d´Hondt	0.2836	
Argentina7	1993	24	128	3.37	d´Hondt	0.2903	
Argentina7	1995	24	128	3.58	d´Hondt	0.2802	

Table 6.7: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Argentina7	1997	24	127	3.65	d´Hondt	0.2794	
Argentina7	1999	24	131	3.15	d´Hondt	0.3085	
Bolivia2	1993	9	130	4.66	S-L	0.4148	
Brazil2	1950	25	304	7.11	d´Hondt	0.3017	
Brazil2	1954	25	326	8.69	d´Hondt	0.2837	
Brazil2	1958	25	326	9.82	d´Hondt	0.2676	
Brazil3	1962	25	404	11.17	d´Hondt	0.2814	
Brazil4	1982	25	479	2.65	d´Hondt	0.4429	
Brazil5	1986	27	495	3.55	d´Hondt	0.414	
Brazil5	1990	27	503	9.79	d´Hondt	0.2968	
Brazil5	1994	27	513	8.53	d´Hondt	0.3161	
Bulgaria1	1991	31	240	4.12	d´Hondt	0.3331	
Bulgaria1	1994	31	240	3.81	d´Hondt	0.343	
Bulgaria1	1997	31	240	3.02	d´Hondt	0.3752	
Cape Verde1	1991	25	79	1.81	d´Hondt		0.3979
Cape Verde2	1995	19	72	2.13	d´Hondt		0.3851
Chile1	1949	28	147	7.05	d´Hondt		0.2323
Chile1	1953	28	147	11.56	d´Hondt		0.1660
Chile1	1957	28	147	8.59	d´Hondt		0.2044
Chile1	1961	28	147	6.44	d´Hondt	0.2101	
Chile1	1965	28	147	4.06	d´Hondt		0.3158
Chile1	1969	28	150	4.92	d´Hondt		0.2887
Chile2	1973	29	152	5.3	d´Hondt		0.2746
Chile3	1993	60	120	6.55	d´Hondt	0.1325	
Chile3	1997	60	120	6.95	d´Hondt	0.1258	
Dominican Rep.1	1966	27	74	2.2	d´Hondt	0.3125	
Dominican Rep.1	1970	27	74	2.9	d´Hondt	0.2564	
Dominican Rep.2	1974	27	91	1.35	d´Hondt	0.4369	
Dominican Rep.2	1978	27	91	2.23	d´Hondt	0.3267	
Dominican Rep.3	1982	27	120	2.76	d´Hondt	0.3042	

Table 6.7: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Dominican Rep.4	1986	30	120	3.19	d´Hondt	0.2674	
Dominican Rep.4	1990	30	120	3.22	d´Hondt	0.2657	
Dominican Rep.4	1994	30	120	2.71	d´Hondt	0.2972	
Dominican Rep.5	1998	30	150	2.73	d´Hondt	0.32	
Finland1	1948	15	200	4.9	d´Hondt	0.3614	
Finland1	1951	15	200	4.96	d´Hondt	0.3674	
Finland2	1954	16	200	4.98	d´Hondt	0.3675	
Finland2	1958	16	200	5.19	d´Hondt	0.3621	
Finland3	1962	15	200	5.86	d´Hondt	0.352	
Finland3	1966	15	200	5.22	d´Hondt	0.3645	
Finland3	1970	15	200	6.17	d´Hondt	0.3428	
Finland3	1972	15	200	5.95	d´Hondt	0.3475	
Finland3	1975	15	200	5.89	d´Hondt	0.3481	
Finland3	1979	15	200	5.74	d´Hondt	0.3467	
Finland3	1983	15	200	5.45	d´Hondt	0.3551	
Finland3	1987	15	200	6.15	d´Hondt	0.3389	
Finland3	1991	15	200	5.89	d´Hondt	0.3436	
Finland3	1995	15	200	5.77	d´Hondt	0.3463	
Finland3	1999	15	200	5.93	d´Hondt	0.3423	
France1	1946	102	544	4.65	d´Hondt		0.2967
France3	1986	96	556	4.66	d´Hondt		0.3063
Guatemala1	1966	22	55	2.83	d´Hondt		0.2886
Guatemala2	1970	22	55	2.83	d´Hondt		0.2886
Israel1	1949	1	120	5.36	d´Hondt	0.4825	
Israel3	1973	1	120	3.81	d´Hondt	0.4886	
Israel3	1977	1	120	5.01	d´Hondt	0.4838	
Israel3	1981	1	120	3.56	d´Hondt	0.4896	
Israel3	1984	1	120	4.3	d´Hondt	0.4866	
Israel3	1988	1	120	5.01	d´Hondt	0.4838	
Israel3	1992	1	120	4.87	d´Hondt	0.4844	

Table 6.7: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Israel3	1996	1	120	5.85	d´Hondt	0.4806	
Israel3	1999	1	120	9.74	d´Hondt	0.4661	
Latvia1	1993	5	100	6.21	S-L	0.4298	
Latvia1	1995	5	100	9.61	S-L	0.3991	
Latvia1	1998	5	100	6.94	S-L	0.4228	
Moldova1	1998	1	104	5.73	d´Hondt	0.4782	
Netherlands1	1946	1	100	4.68	d´Hondt	0.4823	
Netherlands1	1948	1	100	4.98	d´Hondt	0.4809	
Netherlands1	1952	1	100	4.99	d´Hondt	0.4808	
Netherlands2	1956	1	150	4.26	d´Hondt	0.4894	
Netherlands2	1959	1	150	4.46	d´Hondt	0.4884	
Netherlands2	1963	1	150	4.79	d´Hondt	0.4877	
Netherlands2	1967	1	150	6.2	d´Hondt	0.4832	
Netherlands2	1971	1	150	7.09	d´Hondt	0.4805	
Netherlands2	1972	1	150	6.85	d´Hondt	0.4812	
Netherlands2	1977	1	150	3.96	d´Hondt	0.4903	
Netherlands2	1981	1	150	4.56	d´Hondt	0.4884	
Netherlands2	1982	1	150	4.23	d´Hondt	0.4895	
Netherlands2	1986	1	150	3.77	d´Hondt	0.4909	
Netherlands2	1989	1	150	3.9	d´Hondt	0.4905	
Netherlands2	1994	1	150	5.7	d´Hondt	0.4848	
Netherlands2	1998	1	150	5.13	d´Hondt	0.4866	
Norway1	1949	29	150	3.76	d´Hondt	0.3093	
Norway2	1953	20	150	3.53	Mod.S-L	0.3841	
Norway2	1957	20	150	3.44	Mod.S-L	0.3864	
Norway2	1961	20	150	3.59	Mod.S-L	0.3821	
Norway2	1965	20	150	3.9	Mod.S-L	0.3735	
Norway3	1969	20	150	3.61	Mod.S-L	0.3815	
Norway3	1973	19	155	4.63	Mod.S-L	0.3492	
Norway3	1977	19	155	3.85	Mod.S-L	0.368	

Table 6.7: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Norway3	1981	19	155	3.92	Mod.S-L	0.3662	
Norway3	1985	19	155	3.63	Mod.S-L	0.3735	
Portugal1	1976	22	259	4	d´Hondt		0.4059
Portugal2	1979	20	246	3	d´Hondt	0.399	
Portugal2	1980	20	246	2.88	d´Hondt	0.4034	
Portugal2	1983	20	246	3.73	d´Hondt	0.3722	
Portugal2	1985	20	246	4.77	d´Hondt	0.3409	
Portugal2	1987	20	246	2.98	d´Hondt	0.3996	
Portugal2	1991	20	226	2.86	d´Hondt	0.3975	
Portugal2	1995	20	226	3.09	d´Hondt	0.3856	
Portugal2	1999	20	226	3.19	d´Hondt	0.3847	
San Marino1	1993	10	60	3.68	d´Hondt		0.3456
San Marino1	1998	10	60	3.73	d´Hondt		0.3436
Sao Tome1	1991	7	55	2.51	d´Hondt		0.4194
Sao Tome1	1994	7	55	3.2	d´Hondt		0.3906
Sao Tome1	1998	7	55	2.74	d´Hondt		0.4093
Spain1	1977	52	350	4.29	d´Hondt	0.2973	
Spain1	1979	52	349	4.25	d´Hondt	0.2996	
Spain1	1982	52	350	3.18	d´Hondt	0.3436	
Spain1	1986	52	348	3.59	d´Hondt	0.3237	
Spain1	1989	52	348	4.08	d´Hondt	0.3045	
Spain1	1993	52	350	3.5	d´Hondt	0.327	
Spain1	1996	52	350	3.27	d´Hondt	0.3371	
Spain1	2000	52	350	2.99	d´Hondt	0.3509	
Suriname1	1991	10	51	2.69	d´Hondt	0.3407	
Suriname1	1996	10	51	3.62	d´Hondt	0.2888	
Suriname1	2000	10	51	3.77	d´Hondt	0.2819	
Sweden1	1948	28	230	3.35	d´Hondt	0.3687	
Sweden2	1952	28	230	3.28	Mod.S-L	0.4194	
Sweden2	1956	28	231	3.37	Mod.S-L	0.4167	

Table 6.7: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}}_{S_T=\frac{M}{2}}$
Sweden2	1958	28	231	3.31	Mod.S-L	0.4185	
Sweden2	1960	28	232	3.26	Mod.S-L	0.4181	
Sweden2	1964	28	233	3.42	Mod.S-L	0.4116	
Sweden2	1968	28	233	3.18	Mod.S-L	0.4186	
Switzerland1	1947	25	194	5.34	d´Hondt	0.2869	
Switzerland1	1951	25	196	5.09	d´Hondt	0.2951	
Switzerland1	1957	25	196	4.96	d´Hondt	0.2987	
Switzerland1	1959	25	196	5.04	d´Hondt	0.2965	
Switzerland1	1963	25	200	5	d´Hondt	0.3	
Switzerland1	1967	25	200	5.54	d´Hondt	0.2858	
Switzerland1	1971	25	200	6.16	d´Hondt	0.2712	
Switzerland1	1975	25	200	5.78	d´Hondt	0.2825	
Switzerland2	1979	26	200	5.5	d´Hondt	0.2847	
Switzerland2	1983	26	200	5.99	d´Hondt	0.273	
Switzerland2	1987	26	200	6.8	d´Hondt	0.2556	
Switzerland2	1991	26	200	7.34	d´Hondt	0.2453	
Switzerland2	1995	26	200	6.82	d´Hondt	0.2552	
Switzerland2	1999	26	200	5.86	d´Hondt	0.276	
Turkey1	1961	67	450	3.4	d´Hondt		0.3684
Turkey2	1969	67	450	3.3	d´Hondt		0.3725
Turkey2	1973	67	450	4.3	d´Hondt		0.3353
Turkey2	1977	67	450	3.12	d´Hondt		0.3800
Turkey3	1983	83	450	2.85	d´Hondt		0.3727
Turkey4	1995	83	550	6.14	d´Hondt	0.2439	
Turkey5	1999	84	550	6.76	d´Hondt		0.2660
Uruguay1	1946	1	99	3.08	d´Hondt	0.4947	
Uruguay1	1950	1	99	2.65	d´Hondt	0.4968	
Uruguay1	1954	1	99	2.6	d´Hondt	0.497	
Uruguay1	1958	1	99	2.55	d´Hondt	0.4973	
Uruguay1	1962	1	99	2.4	d´Hondt	0.498	

Table 6.7: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for divisor-based electoral systems (cont).

Country	**Year**	**D**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$	$\widehat{V^{nec}_{S_T=\frac{M}{2}}}$
Uruguay1	1966	1	99	2.44	d´Hondt	0.4978	
Uruguay1	1971	1	99	2.74	d´Hondt	0.4963	
Uruguay2	1989	1	99	3.38	d´Hondt	0.4932	
Uruguay2	1994	1	99	3.35	d´Hondt	0.4933	
Uruguay2	1999	1	99	3.12	d´Hondt	0.4945	
Venezuela1	1947	23	110	1.87	d´Hondt		0.4230
Venezuela2	2000	24	165	4.17	d´Hondt		0.3422

Table 6.7 shows aggregated threshold values for 167 democratic elections held in 25 countries in the period 1945-2000. These values suggest similar conclusions to those reached from Table 6.3 above. Lower aggregated threshold values are obtained in those complete electoral systems where district magnitudes are small and where the number of parties is high. The general elections in Chile in 1997 provide the lowest aggregated threshold value in Table 6.7. The complete electoral system used for this election had 120 members of parliament distributed in 60, two-members-districts. The number of competing parties was 6.95, and the electoral formula used to allocate seats was the d'Hondt. The aggregated threshold function to win the majority of the seats in the parliament given this complete electoral system is 0.1258. In other words, to win 60 seats a party must win at least 12.58% of the votes. This relatively low value is not surprising since the design of the electoral system is similar to *winner-takes-all* electoral systems given the small size of districts.

One clear contrast with the aggregated threshold value for the 1997 general election in Chile was the value obtained in the 1962 general election in Uruguay. The complete electoral system used in this case had a single district which elected 99 member and also used the d'Hondt electoral formula. The number of parties competing for those seats was 2.4. The aggregated threshold value obtained given this institutional setting is 0.498. If full proportionality is understood as meaning that the parties' respective share of the vote is the same as their share of the seats, then the complete electoral system used in the 1962 election in Uruguay is pretty close to being an example of this.

As for $S_T = 1$, Table 6.7 does not clearly reveal the effect of the electoral formula. Once again, a simulation must be carried out in order to discover this. Take, once more, the case of the general election held in Latvia in 1993, when 100 seats were contested in 5 districts each with 20 members. The number of parties was 6.21 and the electoral formula used was Sainte-Laguë. This institutional setting produced an aggregated threshold value of 0.4298. To appreciate the effect of a less proportional electoral formula such as d'Hondt, the value of c is changed from 0.5 to 1 and everything else is hold constant. The result that is obtained is 0.3967 which is less proportional than the value obtained when the Sainte-Laguë formula is applied.

In total, 29 cases in Table 6.7 do not include data for the distribution of seats among all districts. A proxy function has been applied in these cases, as explained in Chapter 4. This proxy function has the following form,

$$\widehat{V^{nec}_{S_T}} = \frac{\widehat{S} - 1 + c}{\widehat{M} - 1 + Pc} \tag{6.23}$$

where $\widehat{M}$ and $\widehat{S}$ refer, respectively, to average district magnitude and average seats won as defined above.

When $S_T = \dfrac{M}{2}$ and taking into account the $\widehat{M}$ and $\widehat{S}$ definitions, expression 6.23 takes the following form,

$$\widehat{V^{nec}_{S_T}} = \frac{M + 2D(c-1)}{2(M + D(Pc-1))} \tag{6.24}$$

Remember that the use of this proxy function has some implications that run against the theorems establishing the combination of seats that produces $\min V_{S_T}^{nec}$. The proxy function applied to both quota and divisor-based electoral systems assume that seats are distributed uniformly in all districts and not in the way the theorems establish. Nonetheless, as explained in Chapter 4, when information about $\mathbf{M}_d$ is missing this proxy function should be used to calculate aggregated threshold values.

As the data in this chapter shows, list-proportional representation electoral systems were used in a large proportion of the elections held between 1945-2000. In this chapter I have first described the functioning of both quota- and divisor-based electoral systems. Secondly, proportional representation electoral systems were classified in accordance with the values of the minimum threshold to win the majority of seats in the parliament and to win just 1 seat in that chamber. This reveals the wide range of values that can be found under these electoral systems. Given these variations, interesting questions arise that should be the subject for further research. I have pointed out, for example, that it would be worth studying why different electoral formulae are chosen in different regions of the world. There are a few seminal works that in some way refer to this question (Rokkan 1970; Rogowski 1987; Boix 1999; Benoit 2004). However, the debate still seems to be open. Aggregated threshold values could cast more light on the functioning of these electoral systems and therefore could help answer such key questions. The next chapter covers multi-tier and mixed electoral systems.

Chapter 7

Characterizing Multi-tier and Mixed Electoral Systems

This chapter constitutes an attempt to apply aggregated threshold functions to the complex world of multi–tier and mixed electoral systems. Since these types of electoral systems are becoming progressively more common, to ignore them would mean not taking an increasingly important, if still small, set of electoral systems. The application of aggregated threshold functions to these electoral systems must be considered tentative because a different procedure and different assumptions are needed in order to apply the functions in question.

The rest of the chapter will proceed as follows. First I will discuss the nature and typology of multi-tier systems. As I will show, there are two main types of multi-tier electoral systems: those in which tiers are linked and those in which they are not. In this section, I will explain to which of these types aggregated threshold functions can be applied. After this discussion, data for multi-tier electoral systems will be presented. I will next consider mixed electoral systems. Given the obstacles to agreement about the typology of these electoral systems, I will present the two different, though complementary, characterizations of mixed systems found in the literature. As for multi-tier electoral systems, I will explain which types of electoral systems can be subjected to the application of aggregated threshold functions. Finally, data about mixed member systems will be shown.

7.1 Multi-tier electoral systems.

Multi-tier electoral systems are defined as those in which the seats to be elected are distributed in two or more tiers and the electoral formula employed is the same in all tiers (Golder 2005). The electoral formula used to allocate seats in those tiers may be either proportional or majoritarian. Usually, but not always, these electoral systems have a tier where a fixed number of seats are distributed in multi-member districts- the lower tier (L)- and a higher tier (H) where the remaining seats are distributed in a single district. There are, however, particular cases, such as that of Papua New Guinea, where seats in both tiers are distributed in single-member districts.

Multi-tier electoral systems can be classified depending on whether the tiers are connected or unconnected (Shugart and Wattenberg 2001; Golder 2005). Tiers are connected if the unused votes in one tier are used in another, or when the allocation of seats in one tier is conditional on the seats allocated in another tier. On the contrary, if tiers are unconnected, then seats are allocated in both tiers independently using the share of the vote won by each party (Shvetsova 1999:405).

The number of ballots used in the election is also important. Voters can have either one ballot or as many ballots as tiers exist. In unconnected multi-tier electoral systems, different ballots are normally used for each tier. However, in order to properly apply aggregated threshold functions unconnected multi-tier electoral systems will be treated as if voters had a single ballot. This restriction is based on the assumption that voters are not sophisticated and that they have firm political preferences. In other words, it is assumed that voters always vote for the same party or pre-electoral coalition in both ballots.

Aggregated threshold functions can only be applied to unconnected multi-tier electoral systems. The reasons for this should be familiar by now. When tiers are connected, the estimation of the total number of seats that a party wins depends on actual electoral results and how they produce a concrete allocation of seats among all parties. The distribution of seats in the higher tier depends on the allocation of seats in the lower tier. A prior distribution of seats in the lower tier is, therefore, needed for each political party. Aggregated threshold functions cannot be applied in these cases because the prior allocation of seats for each party cannot be anticipated. It is for this reason that aggregated threshold functions can only be applied to unconnected multi-tier electoral systems.

If tiers are unconnected and if it is assumed that there is only a single ballot, then, a total number of seats for each party can be calculated. This total number of seats is a combination of seats in every tier and must also produce $\min V_{S_T}^{nec}$. The problem here, therefore, is not applying threshold functions to a given number of seats but rather finding out the combination of seats in every tier that produces $\min V_{S_T}^{nec}$. The procedure to find out this number of total seats is explained in the next section.

7.1.1 Data.

In total, 24 democratic elections in 6 different countries have used some kind of unconnected multi-tier electoral system between 1945 and 2000. The regions of the world where this type of electoral systems are most common are Latin America - Ecuador, El Salvador and Guatemala- and Eastern Europe- Poland and Croatia. Institutional stability is a feature of these electoral systems. With the exception of Guatemala, the rest of the countries have systematically used the same electoral system for their democratic processes. Guatemala used three different multi-tier electoral systems between

1990 and 1999.

Before explaining how to calculate the combination of seats in each tier that produces S_T, I introduce, as previously, data about the minimum share of the vote necessary to win 1 seat in the parliament. This data is shown in Table 7.1

Table 7.1: Aggregated threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for unconnected multi-tier electoral systems.

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Ecuador1	1979	L:20 H:1	L:57 H:12	69	6.4	L:Hare H:Hare	0.0130
Ecuador1	1984	L:20 H:1	L:59 H:12	71	10.32	L:Hare H:Hare	0.0080
Ecuador1	1986	L:20 H:1	L:59 H:12	71	11.95	L:Hare H:Hare	0.0069
Ecuador1	1988	L:20 H:1	L:59 H:12	71	8.14	L:Hare H:Hare	0.0102
Ecuador1	1990	L:20 H:1	L:59 H:12	71	7.9	L:Hare H:Hare	0.0105
Ecuador1	1992	L:20 H:1	L:59 H:12	71	7.79	L:Hare H:Hare	0.0106
Ecuador1	1994	L:20 H:1	L:59 H:12	71	7.48	L:Hare H:Hare	0.0111
Ecuador1	1996	L:20 H:1	L:59 H:12	71	6.43	L:Hare H:Hare	0.0129
El Salvador2	1991	L:14 H:1	L:64 H:20	84	3.34	L:Hare H:Hare	0.0149
El Salvador2	1994	L:14 H:1	L:64 H:20	84	3.48	L:Hare H:Hare	0.0143
El Salvador2	1997	L:14 H:1	L:64 H:20	84	3.95	L:Hare H:Hare	0.0126
El Salvador2	2000	L:14 H:1	L:64 H:20	84	3.68	L:Hare H:Hare	0.0135

Table 7.1: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T =$ 1 for unconnected multi-tier electoral systems (cont.).

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Guatemala3	1990	L:23	L:89	116	7.01	L:d´Hondt	0.0302
		H:1	H:27			H:d´Hondt	
Guatemala4	1994	L:23	L:64	80	5.67	L:d´Hondt	0.0483
		H:1	H:16			H:d´Hondt	
Guatemala4	1995	L:23	L:64	80	5.9	L:d´Hondt	0.0478
		H:1	H:16			H:d´Hondt	
Guatemala5	1999	L:23	L:91	113	3.76	L:d´Hondt	0.0438
		H:1	H:22			H:d´Hondt	
Poland2	1993	L:52	L:391	460	9.8	L:d´Hondt	0.0128
		H:1	H:69			H:d´Hondt	
Poland2	1997	L:52	L:391	460	4.59	L:d´Hondt	0.0137
		H:1	H:69			H:d´Hondt	

It should be noted that given the particularity of these electoral systems, calculations to find out aggregated threshold values when $S_T = 1$ must take into account the existing tiers. Hence the value that should be taken into consideration will be the lowest value calculated separately in each tier. As Table 7.1 shows, in the lower tiers seats are distributed in multimember districts. Aggregated threshold functions are applied here in accordance with the theorems given in Chapter 3. Higher-tier seats are distributed in a single district, so aggregated threshold functions are applied. The following example will shed some light on how aggregated threshold values are calculated when $S_T = 1$.

Take for example the general election held in El Salvador in 1991. An unconnected multi-tier electoral system was used in which 64 seats where distributed in 14 districts in the lower tier, and 20 seats in a single district for the higher tier. In total, 84 deputies were elected to parliament. In the lower tier, the smallest district had 3 seats and therefore aggregated threshold functions were applied here by calculating $V_{S_T}^{nec}$ when $S_T = 1$. This produced a value of

0.0047, that is to say, a share of the vote of 0.47%. In the higher tier, this value was 1.49%. The minimum value that the electoral system establishes to win 1 seat is therefore 0.47%, the aggregated threshold value calculated in the smallest district of the lower tier.

The procedure to find out the aggregated threshold value for $V^{nec}_{S_T}$ when $S_T = \dfrac{M}{2}$ is also different and requires some explanation. First, I introduce the data. Aggregated threshold values for unconnected multi-tier electoral systems when $S_T = \dfrac{M}{2}$ are shown in Table 7.2.

Table 7.2: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for unconnected multi-tier electoral systems.

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
Ecuador1	1979	L:20 H:1	L:57 H:12	69	6.4	L:Hare H:Hare	0.2653
Ecuador1	1984	L:20 H:1	L:59 H:12	71	10.32	L:Hare H:Hare	0.2531
Ecuador1	1986	L:20 H:1	L:59 H:12	71	11.95	L:Hare H:Hare	0.2487
Ecuador1	1988	L:20 H:1	L:59 H:12	71	8.14	L:Hare H:Hare	0.2619
Ecuador1	1990	L:20 H:1	L:59 H:12	71	7.9	L:Hare H:Hare	0.2632
Ecuador1	1992	L:20 H:1	L:59 H:12	71	7.79	L:Hare H:Hare	0.2638
Ecuador1	1994	L:20 H:1	L:59 H:12	71	7.48	L:Hare H:Hare	0.2656
Ecuador1	1996	L:20 H:1	L:59 H:12	71	6.43	L:Hare H:Hare	0.273

Table 7.2: Aggregated Threshold values for $V^{nec}_{S_T}$ when $S_T = \frac{M}{2}$ for unconnected multi-tier electoral systems (cont.)

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
El Salvador2	1991	L:14 H:1	L:64 H:20	84	3.34	L:Hare H:Hare	0.3779
El Salvador2	1994	L:14 H:1	L:64 H:20	84	3.48	L:Hare H:Hare	0.3753
El Salvador2	1997	L:14 H:1	L:64 H:20	84	3.95	L:Hare H:Hare	0.3678
El Salvador2	2000	L:14 H:1	L:64 H:20	84	3.68	L:Hare H:Hare	0.3719
Guatemala3	1990	L:23 H:1	L:89 H:27	116	7.01	L:d´Hondt H:d´Hondt	0.201
Guatemala4	1994	L:23 H:1	L:64 H:16	80	5.67	L:d´Hondt H:d´Hondt	0.1975
Guatemala4	1995	L:23 H:1	L:64 H:16	80	5.9	L:d´Hondt H:d´Hondt	0.1916
Guatemala5	1999	L:23 H:1	L:91 H:22	113	3.76	L:d´Hondt H:d´Hondt	0.2888
Poland2	1993	L:52 H:1	L:391 H:69	460	9.8	L:d´Hondt H:d´Hondt	0.2151
Poland2	1997	L:52 H:1	L:391 H:69	460	4.59	L:d´Hondt H:d´Hondt	0.3241

Data in Table 7.2 is estimated using a different algorithm. The procedure used to calculate aggregated values also considers both tiers. The algorithm comprises the following steps.

- For the lower tier, a number of seats is calculated according to the theorems in Chapter 3. This number of seats produces an aggregated threshold value. Or in other words, there is an aggregated threshold value for this number of seats.
- This aggregated threshold value is applied to the higher tier

to find out the number of seats that can be won.

- The number of seats in both the lower and the higher tiers must add up to half of the seats in the parliament.

Consider the following example to see how aggregated threshold values are calculated when $S_T = \frac{M}{2}$. Take the general election in Poland in 1993. In this election 460 seats were up for election using an unconnected multi-tier system. For this purpose, 391 seats were elected in 52 multimember districts using the d'Hondt electoral formula. The size of the districts ranged from 3 to 17. The remaining 69 seats were elected in a single district also applying the d'Hondt electoral formula. The number of parties competing in this election was 9.8. The combination of seats in both tiers that produced the minimum number of votes to win the majority of the seats in the parliament was 213 in the lower tier and 17 in the higher tier. The aggregated threshold value in the lower tier that produced 213 seats was 0.2190. Applying this value to the higher tier, a total number of 17 seats was obtained producing a total of 230 seats when the latter were combined with the lower tier seats. This algorithm constitutes a mechanical way of discovering the number of seats and its related threshold value taking into account both the lower and the higher tier.

As Table 7.2 shows, the values below which it is not possible to win a majority of the seats in the parliament range between the values found in *winner-takes-all* electoral systems and those in PR-electoral systems. The lowest aggregated value is found in the electoral system used during the 1995 general election in Guatemala. The electoral system used in this election was designed to select deputies for 80 seats in two tiers. In the lower tier 64 deputies were elected in 23 multimember districts using the d'Hondt electoral formula. The size of the districts ranged from 1 to 9. The remaining 16 seats were also elected using the d'Hondt electoral formula in a single district. The number of parties competing in this election was 5.9. The minimum value that this electoral system produced to win the majority of the seats in the parliament was 0.1416 or 14.16% of the vote. This value is quite close to those found in *winner-takes-all* electoral systems.

The highest aggregated value, though, reveals something rather different. This value is produced by the electoral system used during the 1991 general election in El Salvador. In this case too, two tiers were used to elect the 84 members of parliament. In the first tier 64 seats were elected in 14 multimember districts using the Hare quota. District magnitudes ranged from 3 to 16 seats. The remaining 20 seats were elected in a single district also using the Hare quota. Given that the electoral formula used in both tiers was quite proportional, the minimum value that this electoral system produced to permit a party to win a majority of seats in parliament was 0.4108. In clear contrast with the value found in the 1995 general election in Guatemala, the value obtained here is closer to those obtained in PR as opposed to *winner-takes-all* electoral systems.

No aggregated threshold data is provided in the case of 6 elections using unconnected multi-tier electoral systems. These elections took place in two different countries: Croatia and Papua New Guinea. The 2000 election in Croatia used an unconnected multi-tier electoral system. The electoral system had not 2 but 3 tiers of seat allocation and complex mechanisms aimed at accommodating all the country's different ethnic groups. The complexity of this electoral system also has a direct effect on the mechanical application of aggregated threshold functions.

The second country for which aggregated threshold data is not provided is Papua New Guinea. Many scholars consider that Papua New Guinea is a country that has had a pure *winner-takes-all* (FPTP) electoral system since it won independence from Australia in 1975 (Reinolds and Reilly 1997: 40). However, this opinion is not shared by others who consider the electoral system used there to be multi-tier (Golder 2005). The reason is that there are two types of legislative members. There are those members elected using a plurality electoral formula in 89 single member districts- these are at the local level. There are also 20 members elected using a plurality electoral formula in 20 single member districts at provincial level. Hence voters have two ballots, one at local level and another at the provincial level (Nohlen, Grotz & Hartman 2001, Golder 2005). This complex system for distributing districts is the main reason why the mechanical application of aggregated threshold functions has not been possible.

Until now, attention has been paid to electoral systems with more than one tiers that use the same electoral formula. In the next section, attention will be paid to a different type of electoral system that also has two or more tiers of seat allocation: mixed electoral systems.

7.2 Mixed electoral systems.

Mixed electoral systems are a rare phenomenon with no unanimous consensus understanding of their typology and definition. Positions in the literature varies from those who simply do not consider this concept (Rae 1967; Mackenzie 1958; Bogdanor 1983; Rose 1983) to others such as Reynolds and Reilly (1997), Massicotte and Blais (1999) or Shugart and Wattenberg (2001) who offer very detailed accounts of this type of electoral system. The lack of consensus even extends to the labels applied to these electoral systems. Whereas Shugart and Wattenberg call them mixed-member electoral systems, others call them hybrid electoral systems (Shvetsova 1999) or simply mixed electoral systems (Massicotte and Blais 1999). I will follow this simple terminology, referring to these electoral systems as mixed electoral systems.

In their work on mixed-member electoral systems Shugart and Wattemberg treat these electoral systems as a subset of multi-tier electoral systems as defined in the previous section. In their view, the particularity of mixed-member electoral systems is that seats are allocated in one tier nominally and using a party list in the other tier. In other words, in one tier voters choose the name of their preferred candidates and seats are allocated to individual candidates; in the other tier, candidates are elected from a list that is submitted by a party or pre-electoral coalition. In accordance with this definition, the nominal tier is usually made up of single-member districts -*winner-takes-all*- that use either a plurality or a majority electoral formula to allocate seats. The list tier, on the contrary, comprises plurinominal districts and uses a PR electoral formula that can be either divisor-based, such as d'Hondt, or a quota-based one, such as Hare (Shugart and Wattenberg 2001:10-13).

Rejecting this view, Massicotte and Blais opt for a very simple definition based on the mechanics of the electoral system. An electoral system is considered to be mixed if two different electoral formulae are applied when allocating seats to a single body. This definition is complemented by a threshold: for an electoral system to be classified as mixed the number of deputies elected under a formula other than the one used for the other deputies must account for at least 5% of the total. As they argue, in accordance with this definition the electoral systems used in Finland or Switzerland are not considered to be mixed, despite the fact that two different electoral formulae are used in both cases. In Finland and Switzerland, the vast majority of seats are allocated using a PR electoral formula, and only a very few using a majoritarian electoral formula (Massicotte and Blais 1999:345).

The definition based on the mechanics of the electoral system proposed by Massicotte and Blais contrasts with the approach taken by others authors such as Reynolds and Reilly (1999) or Shugart and Wattenberg (2001) who base their definitions and classifications on the outcomes of these electoral systems. So, Shugart and Wattenberg distinguish between what they call mixed-member majoritarian (MMM) and mixed-member proportional (MMP) systems. These two different subtypes are defined by whether or not the tiers are connected or unconnected. If tiers are not connected, therefore, the majoritarian boost received by a large party is not corrected by the proportional tier. Therefore, the outcome produced by this subtype of mixed-member electoral system is clearly majoritarian.

If, tiers are connected, then the distortions produced between large and small parties in the nominal tier is corrected by the proportional tier, thereby producing a more proportional outcome (Shugart and Wattenberg 2001:13).

This approach is rejected by Massicotte and Blais, who argue that since it has not been confirmed empirically, it is wrong to assume that plurality-majority formulae invariably produce disproportionate results or that PR formulae always produce more proportional results. As they point out, the distortion effect on large parties produced by plurality-majority electoral formulae may be reduced by using mechanisms such as malapportionment (Massicotte 1995; Lijphart 1994) or by concentrating the majority of the votes of a large party in a few districts (Grofman et al. 1997). While agreeing with these propositions, in this research the mechanical approach will also be adopted given that my primary objective is not to discover if a given type of electoral system is more or less proportional, but rather to uncover its mechanics.

Once the mechanical approach proposed by Massicotte and Blais is accepted, mixed electoral systems can be divided between dependent and independent mixed electoral systems. The distinction depends exclusively on whether the two formulae are used independently or, on the contrary, whether the application of one formula depends on the outcome produced by the other. When the electoral formulae are used independently of each other, then, there are three types of independent mixed electoral systems: coexistence, superposition and fusion (Massicotte and Blais 1999:347-352). Under coexistence mixed systems, elections in one part of the territory are governed by a plurality-majority formula while a PR formula applies in the rest of the territory; this is the system used for the French Senate. Here, in those districts- departments- with fewer than 4 seats, senators are elected using the Two-Round System (TRS). However, in those districts with 5 or more seats, the d'Hondt electoral formula is used. In this system, therefore, electors vote differently depending on the size of the district.

In superposition mixed electoral systems the two electoral formula are applied in two different sets of districts. There are a fixed number of districts where the plurality-majority electoral formula is applied, while the PR electoral formula is used in another fixed number of seats. These electoral systems usually have a fixed number of single-member districts where a plurality-majority electoral formula is applied, as well as a single district from which the rest of the seats are allocated using a list PR electoral formula. So, voters have two different types of representatives: those elected directly in single member districts and those elected in a multimember district using a proportional representation electoral formula. Note that both electoral formulae are applied independently, and PR seats in particular are allocated without any consideration of the results in single member districts. Examples of countries using a superposition mixed electoral system are Ecuador between 1998-2000 or Lithuania between 1992-2000 (Golder 2005).

A variation of the superposition mixed electoral system occurs when the two formulae are combined within one district. This variation is known as the fusion mixed electoral system and is only rarely used. It has been used in the French municipal elections (Massicotte and Blais 1999) as well as in Sri Lanka between 1989 and 2000 (Golder 2005).

When the electoral formulae are used dependently then mixed electoral systems can be conditional or unconditional. Under conditional mixed electoral systems the second electoral formula is only used if the outcome produced by the first electoral formulae does not meet certain conditions. This type of electoral system is not used nowadays and the only examples that can be given to illustrate the application of these systems are the elections held in Italy in 1923 or Romania in 1926. In unconditional mixed electoral systems the two formulae always apply but the way they are combined is dependent because the application of one formula will depend on the outcome produced by the other. The best examples are those cases in which the application of proportional representation electoral formulae corrects the distortions produced by the plurality-majority formula. This is the system used, for example, in Germany (Massicotte and Blais 1999:353-357).

The following example illustrates how dependent and independent mixed electoral systems work. Imagine a parliament with 200 members who are elected using a mixed electoral system with 2 tiers, each of which allocates 100 seats. In the lower tier, deputies are elected by single-member districts, while the remaining 100 seats in the higher tier are distributed in a single district using a perfect PR electoral formula. There are four parties that win a share of the total vote ($V^p(\%)$). Finally, assume the share of the vote that each party wins in the lower results in a known distribution of seats in that tier, $L(s)$, where party A wins 57 seats, party B, 40, party C, 2 and finally, party D wins 1 seat. Table 7.3 shows this information.

Table 7.3: Examples of mixed electoral systems

	Parties			
	A	B	C	D
$V^p(\%)$	47	36	11	6
$L(s)$	57	40	2	1
Superposition				
$H(s)$	47	36	11	6
$S_T(= L(s) + H(s))$	104	76	13	7
Correction				
S'_T	94	72	22	12
$H'(s)(= S'_T - L(s))$	37	32	20	11

Let us suppose first that an independent mixed electoral systems is used; concretely, it is a superposition mixed electoral system. Since the two electoral formulae are applied independently, the total number of seats that each party wins, S_T, is simply the sum of the seats won in the lower tier and in the higher tier. For simplicity's sake, in the higher tier seats are won in accordance with perfect proportionality. In other words, if party A wins 47% of the vote, it also wins 47 seats, and so on for every party.

Let us suppose now that a correction mixed electoral systems is used. Since the electoral formula are dependent in this case, we would first calculate the total number of seats that each party would have won under perfect proportionality according to its share of the vote, S'_T, and assuming that all 200 seats are distributed in a single district. Once this number of seats is known, the number of seats for the higher tier is calculated by subtracting the number of seats that each party was entitled to minus the number of seats won in the lower tier. This is the correction mechanism.

As for multi-tier electoral systems, aggregated threshold functions cannot be applied to all types of mixed electoral systems. Again and for the same reasons and assumptions[1] as in the case of multi-tier electoral systems, only those mixed electoral systems that apply the electoral formulae independently are subjected to the application of aggregated threshold functions. Independent mixed electoral systems allow the calculation of seats in both tiers without assuming any given distribution of votes among all parties. In the next section, I will show how aggregated threshold functions are applied to mixed electoral systems as well as the resulting aggregated threshold values.

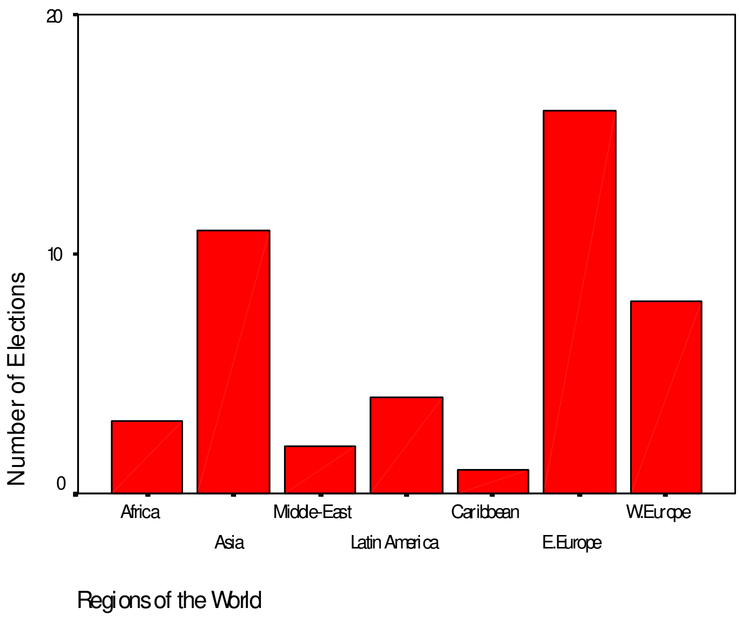
7.2.1 Data.

Independent mixed electoral systems were used in 23 countries between 1945 and 2000. In total, 45 different elections were held in these countries during this period, and 27 different electoral systems were used for them. Most of the independent mixed electoral systems are found in the new democracies that emerged after 1989 in Central and Eastern Europe. This type of electoral systems was also adopted in some Asian countries during the late nineties. Figure 7.1 shows the global distribution of elections that used an

[1] In this sense, it is important to recall that there is a single ballot for both tiers and not a different ballot for each tier. Or, to put this a little differently, it is assumed that in cases with different ballots for each tier, voters always opt for the same political preference in both cases.

independent mixed electoral system between 1945 and 2000.

Figure 7.1: Independent mixed electoral systems used between 1945-2000



Again, I begin the description of the data by considering the minimum conditions to win a minimal political presence in parliament. Aggregated threshold values for independent mixed electoral systems when $S_T = 1$ are shown in Table 7.4

Table 7.4: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for independent mixed electoral systems.

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Albania1	1996	L:115 H:1	L:115 H:25	140	2.73	L:Abs.Major. H:Hare	0.0031
Albania1	1997	L:115 H:1	L:115 H:40	155	2.87	L:Abs.Major. H:Hare	0.0030
Armenia1	1995	L:150 H:1	L:150 H:40	190	4.18	L:Majority H:Hare	0.0015
Armenia2	1999	L:75 H:1	L:75 H:56	131	4.77	L:Majority H:Hare	0.0027
Ecuador2	1998	L:21 H:1	L:105 H:20	125	6.18	L:Plurality H:Hare	0.007
Japan1	1996	L:300 H:11	L:300 H:200	500	3.89	L:Plurality H:d´Hondt	0.0008
Japan1	1998	L:300 H:11	L:300 H:200	500	3.76	L:Plurality H:d´Hondt	0.0008
South Korea2	1988	L:224 H:1	L:224 H:75	299	4.23	L:Plurality H:Hare	0.0010
South Korea3	1992	L:237 H:1	L:237 H:62	299	3.6	L:Plurality H:Hare	0.0011
South Korea4	1996	L:253 H:1	L:253 H:46	299	4.24	L:Plurality H:Hare	0.0009
South Korea5	2000	L:227 H:1	L:227 H:46	273	3.32	L:Plurality H:Hare	0.0013
Kyrgyzstan1	2000	L:45 H:1	L:45 H:15	60	6.6	L:Abs.Major. H:Hare	0.0033
Lithuania1	1992	L:71 H:1	L:71 H:70	141	4.58	L:Abs.Major. H:Hare	0.0030
Lithuania1	1996	L:71 H:1	L:71 H:70	141	7.68	L:Abs.Major. H:Hare	0.0018

Table 7.4: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = 1$ for independent mixed electoral systems (cont.).

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=1}^{nec}$
Lithuania1	2000	L:71 H:1	L:71 H:70	141	7.65	L:Abs.Major. H:Hare	0.0018
Macedonia2	1998	L:85 H:1	L:85 H:35	120	5.01	L:Abs.Major. H:d´Hondt	0.0023
Poland1	1991	L:37 H:1	L:391 H:69	460	13.86	L:Hare H:SL	0.0066
Russia1	1993	L:225 H:1	L:225 H:225	450	3.53	L:Plurality H:Hare	0.0012
Russia1	1995	L:225 H:1	L:225 H:225	450	7.0	L:Plurality H:Hare	0.0006
Russia1	1999	L:225 H:1	L:225 H:225	450	4.3	L:Plurality H:Hare	0.0010
Ukraine2	1998	L:225 H:1	L:225 H:225	450	3.94	L:Plurality H:Hare	0.0011

Let me first make a preliminary observation about the data. Table 7.4 only shows superposition electoral systems. The reasons why fusion electoral systems of the type used in Turkey between 1987 and 1994, or coexistence electoral systems like the one used in Panama between 1989 and 2000 are excluded is basically the inexistence of all the data needed to apply aggregated threshold functions. The rarity of these electoral systems has also made applying these functions problematic. Coexistence electoral systems, as explained above, are characterized by applying one electoral formula to a certain number of districts and a different electoral formula to another set of districts. In reality, these electoral systems have a single tier of seat allocation, to which two different electoral formulae are applied depending on district size. This particularity makes it difficult to apply a reasoning based on the theorems to minimize the aggregated threshold values as introduced in Chapter 3.

In superposition electoral systems aggregated threshold values for $S_T = 1$ are calculated following the same procedure used for unconnected muti-tier electoral systems. As already explained, superposition electoral systems usually, but not always, have a tier of uninominal districts- *winner-takes-all*- and a second tier of seats distributed in a single district. The aggregated threshold values for these electoral systems when $S_T = 1$ is the smallest value in both tiers that is necessary to win 1 seat. Normally, this value is obtained when aggregated threshold functions are applied to single member districts. It is for this reason that the values shown in Table 7.4 are similar to those found for *winner-takes-all* electoral systems.

The range of aggregated threshold values varies from 0.0006 -1995 Russian election- to 0.0033 -2000 Kyrgyzstan election. By looking at these two cases it appears that the larger the parliament and the number of seats in the lower tier, the smaller the aggregated threshold value. One question that deserves some attention is whether the number of seats allocated in each tier has any influence on the aggregated threshold value.

Aggregated threshold values for superposition mixed electoral systems are quite similar to those obtained in *winner-takes-all* systems. Is this also true with respect to the majority of the seats in the parliament? Aggregated threshold values for independent electoral systems when $S_T = \frac{M}{2}$ are given in Table 7.5. Again only superposition electoral systems are included in this table.

Table 7.5: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ for independent mixed electoral systems.

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
Albania1	1996	L:115 H:1	L:115 H:25	140	2.73	L:Abs.Major. H:Hare	0.2046
Albania1	1997	L:115 H:1	L:115 H:40	155	2.87	L:Abs.Major. H:Hare	0.2076
Armenia1	1995	L:150 H:1	L:150 H:40	190	4.18	L:Majority H:Hare	0.1412

Table 7.5: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ for independent mixed electoral systems (cont.).

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}^{nec}_{S_T=\frac{M}{2}}$
Armenia2	1999	L:75 H:1	L:75 H:56	131	4.77	L:Majority H:Hare	0.1563
Ecuador2	1998	L:21 H:1	L:105 H:20	125	6.18	L:Plurality H:Hare	0.0823
Japan1	1996	L:300 H:11	L:300 H:200	500	3.89	L:Plurality H:d´Hondt	0.1820
Japan1	1998	L:300 H:11	L:300 H:200	500	3.76	L:Plurality H:d´Hondt	0.1825
South Korea2	1988	L:224 H:1	L:224 H:75	299	4.23	L:Plurality H:Hare	0.1454
South Korea3	1992	L:237 H:1	L:237 H:62	299	3.6	L:Plurality H:Hare	0.1625
South Korea4	1996	L:253 H:1	L:253 H:46	299	4.24	L:Plurality H:Hare	0.1329
South Korea5	2000	L:227 H:1	L:227 H:46	273	3.32	L:Plurality H:Hare	0.1698
Kyrgyzstan1	2000	L:45 H:1	L:45 H:15	60	6.6	L:Abs.Major. H:Hare	0.0934
Lithuania1	1992	L:71 H:1	L:71 H:70	141	4.58	L:Abs.Major. H:Hare	0.1764
Lithuania1	1996	L:71 H:1	L:71 H:70	141	7.68	L:Abs.Major. H:Hare	0.1131
Lithuania1	2000	L:71 H:1	L:71 H:70	141	7.65	L:Abs.Major. H:Hare	0.1135
Macedonia2	1998	L:85 H:1	L:85 H:35	120	5.01	L:Abs.Major. H:d´Hondt	0.1277
Poland1	1991	L:37 H:1	L:391 H:69	460	13.86	L:Hare H:SL	0.3933

Table 7.5: Aggregated Threshold values for $V_{S_T}^{nec}$ when $S_T = \frac{M}{2}$ for independent mixed electoral systems (cont.).

Country	**Year**	**D**	**Tier**	**M**	**P**	**Formula**	$\mathbf{V}_{S_T=\frac{M}{2}}^{nec}$
Russia1	1993	L:225	L:225	450	3.53	L:Plurality	0.2200
		H:1	H:225			H:Hare	
Russia1	1995	L:225	L:225	450	7.0	L:Plurality	0.1245
		H:1	H:225			H:Hare	
Russia1	1999	L:225	L:225	450	4.3	L:Plurality	0.1880
		H:1	H:225			H:Hare	
Ukraine2	1998	L:225	L:225	450	3.94	L:Plurality	0.2017
		H:1	H:225			H:Hare	

Aggregated threshold values for superposition electoral systems when $S_T = \dfrac{M}{2}$ are calculated on the basis of the reasoning and procedure shown in Chapter 3. Briefly, the minimum value needed to win $S_T = \dfrac{M}{2}$ seats is calculated on the basis of the minimum share of votes that, when applied to both the lower and the higher tier, produces that amount of seats. As shown, in Chapter 3

$$V_{S_{T1}}^{nec} = V_{S_{T2}} \tag{7.1}$$

where S_{T1} and S_{T2} are the number of seats to be won in the lower and the higher tier respectively. Taking into account also that

$$S_{T1} + S_{T2} = \frac{M}{2} \tag{7.2}$$

then the combination of seats that produces $\min V_{S_T}^{nec}$ for superposition electoral systems when $S_T = \dfrac{M}{2}$ and a divisor-based electoral formula is used in the higher tier is given by the following set of equations,

$$\begin{cases} S_{T2} = \dfrac{1}{PM_{T1}} S_{T1}(M_{T2} - 1 + Pc) + 1 - c \\ S_{T1} + S_{T2} = \dfrac{M}{2} \end{cases} \tag{7.3}$$

When a quota-based electoral formula is applied in the higher tier, then the equations to use are,

$$\begin{cases} S_{T2} = \dfrac{S_{T1}P(M_{T2} + n) + PM_{T1}(P - 1 - n)}{P^2 M_{T1}} \\ S_{T1} + S_{T2} = \dfrac{M}{2} \end{cases} \tag{7.4}$$

From these two set of equations the combination of seats in both tiers that produces $\min V_{S_T}^{nec}$ for superposition electoral systems when $S_T = \dfrac{M}{2}$ can be calculated as illustrated in section 2.3.

By looking at Table 7.5 it seems that aggregated threshold values for superposition electoral systems are, again, very similar to the values obtained in *winner-takes-all* electoral systems. This point is also made by Shugart and Wattenberg (2001), who explain how independent mixed electoral system contribute to the boost received by major parties in the lower tier (Shugart and Wattenberg 2001:13). The case of the electoral system used for the general election in Kyrgyzstan in 2000 provides a good example of this.

In 2000 Kyrgyzstan held its second democratic elections after the fall of the communist regime. The electoral system used for this election was a superposition mixed electoral system with 2 tiers of seats allocation. In the first tier 45 seats were elected in single-member districts using a two-round system. In the second tier, 15 seats were distributed in a single districts and seats were allocated through the Hare quota. In total, therefore, 60 seats were elected for the parliament. The number of parties that competed in this election was 6.6. The aggregated threshold value calculated to win the majority of the seats in the parliament for this electoral system is 0.0934. In other words, parties had to obtain at least 9.34% of the vote in order to win this number of seats.

The low value obtained under this electoral system and in all other electoral systems seems to substantiate Shugart and Wattenberg's assertion that this type of electoral system produces rather majoritarian electoral outcomes. This can be seen if the values in Table 7.5 are contrasted with the values shown for *winner-takes-all* electoral systems. However, note that there is a country that have used a mixed electoral system whose value is closer to those found under PR systems. The electoral system used in Poland in 1991 shows values closer to 40% of the vote. Curiously, this electoral system does not have uninominal districts at the lower level but rather use multimember districts. Concretely in the electoral system used in the 1991 elections in Poland, 391 seats were elected in 37 multinominal districts. The average magnitude was about 10.5 seats and district magnitude ranged from 3 to 17. This probably explains why the value obtained is higher than the values produced in those electoral systems using single-member districts at the lower level.

Aggregated values for multi-tier and mixed electoral systems should be seen as tentative. While I have tried to apply aggregated threshold functions as rigorously as possible, the complexity of the mechanics of these electoral systems means the values should be treated with caution. One point in favour of these values is that they seem coherent with the literature. The relationship between superposition electoral system and majoritarian outcomes as expected by Shugart and Wattenberg points in this direction. However, multi-tier and mixed electoral systems deserve further attention, and new methods or even new functions may be needed. Regarding these fashionable electoral systems the field of research is broad. How could aggregated threshold values be calculated when two ballots are used? How could these values also be calculated when tiers are connected or electoral formulae dependent? Future research on these questions would be welcome and its results illuminating.

Chapter 8

Conclusions

This thesis has pursued two main goals. First, to elaborate a measure capable of capturing the mechanical functioning of any electoral system. Second, it has presented a characterization of most of the electoral systems used between 1945 and 2000. The two goals are connected since the former is the means used to characterize all referred electoral systems.

The first part of the thesis focused on calculating aggregated threshold functions. These functions show the necessary and sufficient proportion of votes to win a given number of seats nationwide under any complete electoral system. Furthermore, this dissertation has shown how these functions are not only logically and formally defined, but they also produce values that are consistent with electoral results. Aggregated threshold functions, therefore, have a strong capacity to calculate the necessary and sufficient proportions of the vote to win any number of seats distributed in specific ways among all districts in any complete electoral system. In the second part of this study, electoral systems have been characterized on the basis of the minimum share of the vote necessary to win a given number of seats. This condition establishes the minimum threshold that any party must achieve in a specific complete electoral system in order to win a given number of seats that are particularly distributed among all districts. If a party fails to reach this threshold it will have no chance of winning that number of seats. Electoral systems are characterized according to this minimum criterion and around two values. First, the minimum proportion of votes required to win just one seat nationwide. Second, the minimum proportion of votes required to win half the seats in parliament, that is, an absolute majority.

One of the main results of this thesis is precisely the method used to obtain these minimum values. Given that seats are distributed in districts, the optimization method seeks to identify the combination of seats that produces the lowest share of the vote to win either one seat or half the seats in parliament. As for the optimization of the aggregated threshold functions for one seat, the combination of seats where the minimum value is obtained depends on the type of electoral formula that is used. With respect to quota-based electoral formulae, when the complete electoral system uses the Hare quota ($n = 0$), the single seat can be won in any district since the value to win that seat is the same in all districts. However, when the Droop quota is used ($n = 1$), then that seat must be won in the smallest district. Under divisor-based electoral systems, however, only when the seat is won in the smallest district is the aggregated threshold function minimized, no matter the value of c.

In order to optimize the aggregated threshold function for half the seats in the parliament we must take into consideration not only the different families of electoral formulae but also how seats are distributed among all districts. First I analyzed systems in which seats are distributed unequally among all districts, that is to say, districts magnitudes are different. For divisor-based electoral formula, the combination of seats that produces the minimum proportion of votes to win half of the seats in the parliament when the d'Hondt electoral formula ($c = 1$) is used, is one in which seats are distributed in all small districts. Seats must be won from the smallest to the biggest district until the total number of seats is reached. When the divisor-based electoral formula used is Sainte-Laguë ($c = 0.5$), the combination of seats that minimizes the aggregated threshold functions is one in which all seats must be distributed among all districts. Seats must be won in all small districts, however, at least one seat must be won in all districts, no matter their size. This latter distribution of seats also minimizes aggregated threshold functions when the electoral formula used is based on a quota and, again, district magnitudes are unequal.

Seats can, however, be distributed in districts of equal size. When this is the case, and a divisor-based electoral formula is used, the combination of seats that minimizes aggregated threshold functions have the following particularities. When the electoral formula used is d'Hondt, then no matter how seats are distributed all possible distributions of them produce the same result of winning half of the seats in the parliament. In contrast, when the Sainte-Laguë formula is used instead, all possible combinations of seats that lead to winning the majority of the seats in the parliament produce the same value, however there must be at least one seat in each district. This is also the case when a quota-based electoral formula is used to allocate seats.

Electoral systems can be characterized in function of these criteria. In this sense, all electoral systems can be located on a continuum showing the minimum proportion of votes required to win one seat or a parliamentary majority. Along this continuum, complete electoral systems can be positioned according to their proximity to the ideal point of perfect proportionality. This way of characterizing electoral systems enables us to identify which institutional setting produces higher or lower aggregated values. *Winner-takes-all* complete electoral systems mainly depend on the number of parties. The larger the number of parties, the lower the minimum proportion of votes required to win a majority of seats in parliament. These electoral systems also produce values that are far removed from the point of perfect proportionality. This thesis shows that in winner-takes-all systems, when just two parties compete, the majority of seats in the parliament, in other words 50% of the seats, can only be won by a party obtaining at least 25% of the vote.

List proportional representation electoral systems behave rather differently. Obviously, holding all other variables equal, more proportional electoral formulae such as Sainte-Laguë or Hare produce values closer to the point of perfect proportionality than more majoritarian formulae such as d'Hondt or Droop. However, this thesis shows how the proximity to perfect proportionality does not depend exclusively on the type of electoral formula used. The importance of assembly size, the number of parties, and the number of districts and their sizes must also be taken into consideration. In this sense, it should be noted that, holding the other variables constant, when all districts are of the same size the aggregated threshold value for half of the seats in parliament increases as the number of districts decreases. This is also the case when districts size varies. In this case, the minimum proportion of the vote required to win half the seats in the parliament decreases as the number of unequal districts increases, holding the rest of the variables constant[1]. To sum up all these ideas, list proportional representation electoral systems with a single district, larger assemblies and a small number of parties produce the values closest to the point of perfect proportionality. However, when the number of districts is large the aggregated threshold values produced by those complete electoral systems are much less close to the point of perfect proportionality.

Aggregated threshold values for multi-tier and mixed electoral systems are tentative. This is so because of the particular institutional design of these systems. Aggregated threshold functions are applied to these electoral systems following mechanical algorithms and taking into account the minimizing criteria when they can be applied. The results obtained are, nonetheless, consistent with their mechanical functioning. So, independent mixed-electoral systems with one tier of uninominal districts and a second tier with a single district produce values quite distant from the point of perfect proportionality and closer to those produced by winner-takes-all complete electoral systems. However, in the case of unconnected

[1] Appendix A offers a more formalized explanation of the relationship between the number of districts and $V^{nec}_{S_T=\frac{M}{2}}$.

multi-tier complete electoral systems with a first tier consisting of multimember districts and a second tier made up of a single district, the values produced are quite similar to those found for list proportional representation electoral systems; this is also the case when this same setting is adopted in mixed electoral systems.

As noted in the Introduction, the main goal of this dissertation has been to create an instrument to characterize any electoral system. Aggregated threshold functions should, therefore, be considered to be useful tools for this purpose. Given that the electoral system is an important variable in many areas of political science research, a measure of the type elaborated here should be of some use to the discipline. Some of the questions posited here, along with those referred to in the existing literature, constitute ideal scenarios in which to test the values presented in this study.

Appendix A

Optimizing the number of Districts (D)

Politicians may want to be able to discover the conditions which optimize each variable in aggregated threshold functions. An important variable is the number of districts (D). The question is, holding all other variables constant, what is the effect of the number of districts on aggregated threshold functions? The following theorems answer this question.

Theorem 7 *Let D be districts of constant magnitude, the value of $V_{S_T}^{nec}$ for $S_T = \frac{M}{2}$ decreases as D increases, holding the rest of the variables equal.*

Proof. The proof will proceed as follows. First, I will prove that theorem 7 is true for any complete electoral system that uses a d'Hondt electoral formula. Second, I will prove that this holds for any complete electoral system that uses a Sainte-Laguë electoral formula[1].

[1] The same reasoning can be applied to quota-based electoral formulae. For simplicity's sake and in order to avoid lengthy proofs, I will only refer to divisor-based electoral formulae.

Let λ be a positive integer such that

$$M = \lambda D \tag{A.1}$$

In other, words, λ represents the size of all districts in vector $\mathbf{M}_d$. The value of λ ranges from 1 to M,

$$1 \leq \lambda \leq M \tag{A.2}$$

Having defined the distribution of district magnitudes, be a complete electoral systems with an assembly size, M, a number of districts, D, a vector containing all district magnitudes, $\mathbf{M}_d$ where $M_d = M_D$, a number of competing parties, P, and a divisor-based electoral formula such that $c = 1$. Given this institutional setting, let $D = 1$; the minimum share of votes to win $S_T^D = \frac{M}{2}$, $V_{S_T^D}^{nec}$ in this case is

$$V_{S_T^D}^{nec} = \frac{S_T^D - 1 + c}{M - 1 + Pc} \tag{A.3}$$

Since $S_T^D = \frac{M}{2}$ this expression can be simplified as

$$V_{S_T^D}^{nec} = \frac{M - 2 + 2c}{2(M - 1 + Pc)} \tag{A.4}$$

Furthermore, since $c = 1$, this expression can be simplified as

$$V_{S_T^D}^{nec} = \frac{M}{2(M - 1 + P)} \tag{A.5}$$

Now suppose another complete electoral system where $D > 1$, the other variables remaining constant. From theorem 4 we know that for any complete electoral system that uses a d'Hondt electoral formula and where all districts are of equal magnitude, the minimum share of the vote required to win $S_T^{\frac{M}{\lambda}} = \frac{M}{2}$ is the same for all possible combination of seats that produce $S_T^{\frac{M}{\lambda}}$. Taking this into account, under this complete electoral system, $V^{nec}_{S_T^{\frac{M}{\lambda}}}$ can be calculated as follows,

$$V^{nec}_{S_T^{\frac{M}{\lambda}}} = \frac{D}{2}\left[\frac{\lambda}{M}\left(\frac{\lambda - 1 + c}{\lambda - 1 + Pc}\right)\right] \tag{A.6}$$

Using equation A.1 above, this expression can be simplified as

$$V^{nec}_{S_T^{\frac{M}{\lambda}}} = \frac{\lambda - 1 + c}{2\left(\lambda - 1 + Pc\right)} \tag{A.7}$$

For theorem 7 to be true it must be the case that the value of $V^{nec}_{S_T^D}$ when $D = 1$ is higher than the value of $V^{nec}_{S_T^{\frac{M}{\lambda}}}$ when $D > 1$.

$$\frac{M}{2(M - 1 + Pc)} \geq \frac{\lambda - 1 + c}{2\left(\lambda - 1 + Pc\right)} \tag{A.8}$$

Reducing both terms we get

$$\frac{M}{M - 1 + P} \geq \frac{\lambda}{\lambda - 1 + P} \tag{A.9}$$

Solving for λ, the following result is obtained,

$$M \geq \lambda \tag{A.10}$$

Since $M = \lambda D$, then expression A.10 is reduced to

$$D \geq 1 \tag{A.11}$$

which is always true by definition.

Let us now suppose that the electoral formula used in this set of complete electoral systems is Sainte-Laguë, $c = 0.5$. When $D = 1$, the value of $V^{nec}_{S_T^D}$ for $S_T^D = \dfrac{M}{2}$ in this complete electoral system is given by the following expression,

$$V^{nec}_{S_T^D} = \frac{M - 1}{2M - 2 + P} \tag{A.12}$$

As above, suppose now that $D \geq 1$, holding all other variables constant. Also suppose that in accordance with theorem 4, the S_T seats are won in equal numbers in each district. In other words, the number of seats won in each district size λ equals $\frac{M}{2D}$. The value $V^{nec}_{S_T^{\frac{M}{\lambda}}}$ when $S_T^{\frac{M}{\lambda}} = \frac{M}{2}$ for this complete electoral system is, therefore,

$$V^{nec}_{S_T^{\frac{M}{\lambda}}} = D\left[\frac{\lambda}{M}\left(\frac{\frac{M}{2D} - 1 + c}{\lambda - 1 + Pc}\right)\right] \tag{A.13}$$

Since $c = 0.5$ and since $M = \lambda D$, this expression can be simplified to

$$\frac{M - D}{D(2\lambda - 2 + P)} \tag{A.14}$$

For theorem 7 to be true the following must hold

$$V^{nec}_{S_T^D} \geq V^{nec}_{S_T^{\frac{M}{\lambda}}} \tag{A.15}$$

Formally,

$$\frac{M - 1}{2M - 2 + P} \geq \frac{M - D}{D(2\lambda - 2 + P)} \tag{A.16}$$

Again, solving for D we obtain,

$$D \geq 1 \tag{A.17}$$

which is always true by definition. ■

Theorem 8 *Given a set of complete electoral systems where the vector* $\mathbf{M}_d$ *is made of uneven district magnitudes, the value of* $V^{nec}_{S_T}$ *for* $S_T = \frac{M}{2}$ *in any of these complete electoral systems decreases* *as the number of districts, D, increases, holding the rest of the variables constant.*

Proof. As in the previous proof, I will start by focusing first on a d'Hondt electoral formula ($c = 1$) and after that a proof will be provided for a Sainte-Laguë electoral formula ($c = 0.5$). Let there be two different complete electoral systems. The only difference between them lies in the size of vector $\mathbf{M}_d$. In the first complete electoral system, the cardinality of vector $\mathbf{M}_d^D$ equals the number of districts, D, $\mid \mathbf{M}_d^D \mid = D$. However, in the second complete electoral system the cardinality of vector $\mathbf{M}_d^{D+\Delta}$ is $\mid \mathbf{M}_d^{D+\Delta} \mid = D + M_D - 1$, where $\Delta \geq 1$

Vector $\mathbf{M}_d^D$ is a $1xD$ vector with the following elements

$$\mathbf{M}_d^D = \left[\begin{array}{cccc} M_1 & M_2 & ... & M_D \end{array} \right] \tag{A.18}$$

where $M_1 > M_2 > ... > M_D$

Vector $\mathbf{M}_d^{D+\Delta}$ is a $1x(D + M_D - 1)$ vector with the following elements

$$\mathbf{M}_d^{D+\Delta} = \left[\begin{array}{ccccc} M_1 & M_2 & ... & 1 & 1 \end{array} \right] \tag{A.19}$$

Vectors $\mathbf{M}_d^{D+\Delta}$ and $\mathbf{M}_d^D$ differ in that the former contains as many single-member districts as the size of $M_D \in \mathbf{M}_d^D$ or as the size of the next M_d member of $\mathbf{M}_d^D$ which is larger than 1.

In order for theorem 8 to be true, the minimum number of votes to win half of the seats in the parliament in the first complete electoral system, $V^{nec}_{S_T^D}$, must be higher that the number of votes to win half of the seats in the parliament under the second complete electoral system, $V^{nec}_{S_T^{D+\Delta}}$. Formally,

$$V^{nec}_{S_T^D} \geq V^{nec}_{S_T^{D+\Delta}} \tag{A.20}$$

Since vectors $\mathbf{M}_d^D$ and $\mathbf{M}_d^{D+\Delta}$ only differ in how element M_D is conceived, $V^{nec}_{S_T^D}$ and $V^{nec}_{S_T^{D+\Delta}}$ can be reduced to the consideration of the M_D element.

$$\frac{M_D}{M}\left(\frac{M_D - 1 + c}{M_D - 1 + Pc}\right) \geq M_D\left(\frac{1}{MP}\right) \tag{A.21}$$

considering the d'Hondt electoral formula, $c = 1$.

Expression A.21 can be reduced to

$$\frac{M_D^2}{M_D - 1 + P} \geq M_D\frac{1}{P} \tag{A.22}$$

Solving this inequality, the following is obtained

$$M_D\left(M_D P - M_D + 1 - P\right) \geq 0 \tag{A.23}$$

Two conclusions can be drawn at this point. The first is that if inequality A.23 is true, then $M_D \geq 0$, which is always true since any district must contain at least one seat to be distributed among all competing parties. The second conclusion depends on solving the following inequality

$$M_D P - M_D + 1 - P \geq 0 \tag{A.24}$$

Solving for P we obtain

$$P \geq 1 \tag{A.25}$$

which is always true by definition. If this is true for any M_D element, then by induction, theorem 8 must be true.

When the Sainte-Laguë electoral formula is used a similar reasoning can be developed to prove this theorem. According to theorem **3,** when this electoral formula is used, seats must be distributed first among the smallest districts and then progressively among the larger districts until all the S_T seats have been assigned. However, all districts, no matter their size, must have at least one seat. Given that vectors $\mathbf{M}_d^D$ and $\mathbf{M}_d^{D+\Delta}$ are equal except in terms of how the smallest element of $\mathbf{M}_d^D$, i.e. M_D, is distributed, theorem 3 will not apply since the seats in district M_D will always be assigned. The proof of this theorem is exactly the same as for the d'Hondt formula, but applying the value for Sainte-Laguë instead. Formally,

$$\frac{M_D}{M}\left(\frac{M_D - 1 + \frac{1}{2}}{M_D - 1 + \frac{P}{2}}\right) \geq M_D\left(\frac{1}{MP}\right) \tag{A.26}$$

which can be simplified to

$$\frac{M_D(2M_D - 1)}{2M_D - 2 + P} \geq \frac{M_D}{P} \tag{A.27}$$

Inequality A.27 can be reduced to

$$M_D P - P \geq M_D - 1 \tag{A.28}$$

and solving for P, the following is obtained

$$P \geq 1 \tag{A.29}$$

which is always true by definition. If this is true for any M_D, then by induction, when the Sainte-Laguë electoral formula is used, theorem 8 is also true. ■

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Norway: John-Erik Agotnes (private communication)

Peru: Oficina Nacional de Procesos Electorales (ONPE) (private communication)

Portugal: Comissao Nacional de Eleiçoes

Spain: Ministerio de Interior

Sweden: Statistica Centralbyran (Stockolm)

Switzerland: Swiss Federal Statistical Office

General Web Sites

Elections Around the World: www.electionworld.org

Election Archive: http://psephos.adam-carr.net

Election Resources on the Internet: www.electionresources.org

Elections and Electoral Systems Around the World:
www.psr.keele.ac.uk/election.htm

International Foundation for Election Systems: www.ifes.org

International Institute for Democracy and Electoral Assistance:
www.idea.int

Lijphart Elections Archive: http://dodgson.ucsd.edu/lij/

NCSEER Post-communist Elections Project:
www.princeton.edu/~jtucker/pcelections.html

Political Database of the Americas: www.georgetown.edu/pdba/

Political Transformation and the Electoral Process in Post-communist Europe:
www.essex.ac.uk/elections